CHAPTER 7
THE FOURIER TRANSFORM

Up to this point, the main theme of this book has been the theory and application of infinite series expansions involving various orthonormal sets of functions. We now turn to the study of integral transforms, a different but related collection of techniques for analyzing functions and solving differential equations. We begin with the Fourier transform, which provides a way of expanding functions on the whole real line $\mathbb{R} = (-\infty, \infty)$ as (continuous) superpositions of the basic oscillatory functions $e^{i\xi x}$ ($\xi \in \mathbb{R}$) in much the same way that Fourier series are used to expand functions on a finite interval. To provide some motivation, let us perform a few formal calculations.

Suppose that f is a function on $\mathbb{R}$. For any $l > 0$ we can expand f on the interval $[-l, l]$ in a Fourier series, and we wish to see what happens to this expansion as we let $l \to \infty$. To this end, we write the Fourier expansion as follows: For $x \in [-l, l]$,

$$f(x) = \frac{1}{2l} \sum_{-\infty}^{\infty} c_{n,l} e^{i\pi n x/l}, \qquad c_{n,l} = \int_{-l}^{l} f(y) e^{-i\pi n y/l} \, dy.$$

Let $\Delta\xi = \pi/l$ and $\xi_n = n\Delta\xi = n\pi/l$; then these formulas become

$$f(x) = \frac{1}{2\pi} \sum_{-\infty}^{\infty} c_{n,l} e^{i\xi_n x} \Delta\xi, \qquad c_{n,l} = \int_{-l}^{l} f(y) e^{-i\xi_n y} \, dy.$$

Let us suppose that $f(x)$ vanishes rapidly as $x \to \pm\infty$; then $c_{n,l}$ will not change much if we extend the region of integration from $[-l, l]$ to $(-\infty, \infty)$:

$$c_{n,l} \approx \int_{-\infty}^{\infty} f(y) e^{-i\xi_n y} \, dy.$$

This last integral is a function only of ξ_n, which we call $\widehat{f}(\xi_n)$, and we now have

$$f(x) \approx \frac{1}{2\pi} \sum_{-\infty}^{\infty} \widehat{f}(\xi_n) e^{i\xi_n x} \Delta\xi \qquad (|x| < l).$$

This looks very much like a Riemann sum. If we now let $l \to \infty$, so that $\Delta\xi \to 0$, the $\approx$ should become $=$ and the sum should turn into an integral, thus:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\xi)e^{i\xi x}\,d\xi, \quad \text{where } \widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i\xi x}\,dx. \tag{7.1}$$

These limiting calculations are utterly nonrigorous as they stand; nonetheless, the final result is correct under suitable conditions on f, as we shall prove in due course. The function $\widehat{f}$ is called the **Fourier transform** of f, and (7.1) is the **Fourier inversion theorem**.

Before proceeding, we establish a couple of notational conventions. We shall be dealing with functions defined on the real line, and most of our integrals will be definite integrals over the whole line. Accordingly, we shall agree that an integral sign with no explicit limits means the integral over $\mathbf{R}$ (and not an indefinite integral):

$$\int f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx.$$

Moreover, L^2 will mean $L^2(\mathbf{R})$, the space of square-integrable functions on $\mathbf{R}$.

We also introduce the space $L^1 = L^1(\mathbf{R})$ of **(absolutely) integrable** functions on $\mathbf{R}$:

$$L^1 = \left\{ f : \int |f(x)|\,dx < \infty \right\}.$$

(Here, as with L^2, the integral should be understood in the Lebesgue sense, but this technical point will not be of any great concern to us.) We remark that L^1 is not a subset of L^2, nor is L^2 a subset of L^1. The singularities of a function in L^1 (that is, places where the values of the function tend to ∞) can be somewhat worse than those of a function in L^2, since squaring a large number makes it larger; on the other hand, functions in L^2 need not decay as rapidly at infinity as those in L^1, since squaring a small number makes it smaller. For example, let

$$f(x) = \begin{cases} x^{-2/3} & \text{if } 0 < x < 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$g(x) = \begin{cases} x^{-2/3} & \text{if } x > 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then f is in L^1 but not in L^2, whereas g is in L^2 but not in L^1. (The easy verification is left to the reader.) However, we have the following useful facts:

(i) If $f \in L^1$ and f is bounded, then $f \in L^2$. Indeed,

$$|f| \le M \implies |f|^2 \le M|f| \implies \int |f(x)|^2 dx \le M \int |f(x)|\,dx < \infty.$$

(ii) If $f \in L^2$ and f vanishes outside a finite interval $[a, b]$, then $f \in L^1$. This follows from the Cauchy-Schwarz inequality:

$$\int |f(x)|\,dx = \int_a^b 1 \cdot |f(x)|\,dx \le (b-a)^{1/2}\left(\int_a^b |f(x)|^2 dx\right)^{1/2} < \infty.$$

7.1 Convolutions

Before studying the Fourier transform, we need to introduce the convolution product of two functions, a device that will be very useful both as a theoretical tool and in applications. This idea may seem a bit mysterious to the reader who has never seen it before; we shall make a few comments on its meaning following Theorem 7.2, but a fuller appreciation of its significance can best be achieved by seeing how it arises throughout the course of this chapter.

If f and g are functions on $\mathbb{R}$, their **convolution** is the function $f * g$ defined by

$$f * g(x) = \int f(x-y)g(y)\,dy, \tag{7.2}$$

provided that the integral exists. Various conditions can be imposed on f and g to ensure that the integral will be absolutely convergent for all $x \in \mathbb{R}$, for example:

(i) If $f \in L^1$ and g is bounded (say $|g| \le M$), then

$$\int |f(x-y)g(y)|\,dy \le M \int |f(x-y)|\,dy = M \int |f(y)|dy < \infty.$$

(ii) If f is bounded (say $|f| \le M$) and $g \in L^1$, then

$$\int |f(x-y)g(y)|\,dy \le M \int |g(y)|\,dy < \infty.$$

(iii) If f and g are both in L^2, then by the Cauchy-Schwarz inequality,

$$\int |f(x-y)g(y)|\,dy \le \sqrt{\int |f(x-y)|^2 dy}\sqrt{\int |g(y)|^2 dy} = \|f\|\,\|g\| < \infty.$$

(iv) If f is piecewise continuous and g is bounded and vanishes outside a finite interval $[a, b]$, then $f * g(x)$ exists for all x, since the function $y \to f(x-y)$ is bounded on $[a, b]$ for any x.

(v) It can be shown that if f and g are both in L^1, then $f * g(x)$ exists for "almost every" x, i.e., for all x except for some set having Lebesgue measure zero; moreover, $f * g \in L^1$. See Folland [25], §8.1, or Wheeden-Zygmund [56], §9.1.

This list can be extended. In what follows we assume implicitly that the functions we mention satisfy appropriate conditions so that all integrals in question are absolutely convergent. The reader may supply specific hypotheses at will; it would often be quite tedious to list all possible ones.

We now investigate the basic algebraic and analytic properties of convolutions.

Theorem 7.1. *Convolution obeys the same algebraic laws as ordinary multiplication:*

(i) $f * (ag + bh) = a(f * g) + b(f * h)$ *for any constants* a, b;

(ii) $f * g = g * f$;

(iii) $f * (g * h) = (f * g) * h$.

Proof: (i) is obvious since integration is a linear operation. For (ii), make the change of variable $z = x - y$:

$$f * g(x) = \int f(x-y)g(y)\,dy = \int f(z)g(x-z)\,dz = g * f(x).$$

For (iii), use (ii) and interchange the order of integration:

$$(f * g) * h(x) = \int f * g(x-y)h(y)\,dy = \iint f(z)g(x-y-z)h(y)\,dz\,dy$$
$$= \iint f(z)g(x-z-y)h(y)\,dy\,dz = \int f(z)g * h(x-z)\,dz = f * (g * h)(x). \quad ∎$$

Theorem 7.2. *Suppose that f is differentiable and the convolutions $f * g$ and $f' * g$ are well-defined. Then $f * g$ is differentiable and $(f * g)' = f' * g$. Likewise, if g is differentiable, then $(f * g)' = f * g'$.*

Proof: Just differentiate under the integral sign:

$$(f * g)'(x) = \frac{d}{dx}\int f(x-y)g(y)\,dy = \int f'(x-y)g(y)\,dy = f' * g(x).$$

Since $f * g = g * f$, the same argument works with f and g interchanged. ∎

We emphasize that in Theorem 7.2 one can throw the derivative in $(f * g)'$ onto either factor. Thus $f * g$ is at least as smooth as either f or g, even when the other factor has no smoothness properties.

Let us pause to make a few remarks that may shed some light on the meaning of convolutions. In the first place, let us think of the convolution integral as a limit of Riemann sums,

$$\int f(x-y)g(y)\,dy \approx \sum f(x-y_j)g(y_j)\Delta y_j.$$

The function $f_j(x) = f(x - y_j)$ is the function f translated along the x-axis by the amount y_j, so the sum on the right is a linear combination of translates of f with coefficients $g(y_j)\Delta y_j$. We can therefore think of $f * g$ as a continuous superposition of translates of f; and since $f * g = g * f$, it is also a continuous superposition of translates of g.

Second, convolutions may be interpreted as "moving weighted averages." We recall that the average value of a function f on the interval $[a, b]$ is defined to be $(b-a)^{-1}\int_a^b f(y)\,dy$. More generally, the weighted average of f on $[a, b]$ with respect to a nonnegative weight function w is

$$\frac{\int_a^b f(y)w(y)\,dy}{\int_a^b w(y)\,dy}.$$

Suppose now that g is nonnegative and $\int g(y)\,dy = 1$. If we write $f * g(x)$ as $\int f(y)g(x-y)\,dy$, we see that $f * g(x)$ is the weighted average of f (on the whole line) with respect to the weight function $w(y) = g(x-y)$. If $g(x) = 0$ for $|x| > a$ then $g(x-y) = 0$ for $|x-y| > a$, so $f * g(x)$ is a weighted average of f on the interval $[x-a, x+a]$. In particular, if

$$g(x) = \begin{cases} (2a)^{-1} & \text{if } -a < x < a, \\ 0 & \text{otherwise,} \end{cases}$$

then

$$f * g(x) = \frac{1}{2a} \int_{x-a}^{x+a} f(y)\, dy,$$

which is the (ordinary) average of f on the interval $[x-a, x+a]$.

One respect in which convolution does not resemble ordinary multiplication is that whereas $f \cdot 1 = f$ for all f, there is no function g such that $f * g = f$ for all f. (The Dirac "δ-function" does the job, but it is not a genuine function; we shall discuss it in Chapter 9.) However, we can easily find sequences $\{g_n\}$ such that $f * g_n$ converges to f as $n \to \infty$. The intuition is provided by the remarks of the preceding paragraph: If $g(x)$ vanishes (or at least is negligibly small) outside an interval $|x| < a$, then $f * g(x)$ will be a weighted average of the values of f on the interval $[x-a, x+a]$, and if a is very small this should be approximately $f(x)$.

To be precise, suppose $g \in L^1$, and for $\epsilon > 0$ let

$$g_\epsilon(x) = \frac{1}{\epsilon} g\left(\frac{x}{\epsilon}\right). \tag{7.3}$$

That is, g_ϵ is obtained from g by compressing the graph in the x-direction by a factor of ϵ and simultaneously stretching it in the y direction by a factor of $1/\epsilon$. (We are thinking of the case $\epsilon < 1$; if $\epsilon > 1$ the words *compressing* and *stretching* should be interchanged. See Figure 7.1.) As $\epsilon \to 0$ the graph of g_ϵ becomes a sharp spike at $x = 0$, but the area under the graph remains constant:

$$\int g_\epsilon(x)\, dx = \int g\left(\frac{x}{\epsilon}\right) d\left(\frac{x}{\epsilon}\right) = \int g(y)\, dy.$$

More generally, the substitution $x = \epsilon y$ yields

$$\int_a^b g_\epsilon(x)\, dx = \int_{a/\epsilon}^{b/\epsilon} g(y)\, dy. \tag{7.4}$$

With this in mind, we can state a precise theorem.

Theorem 7.3. *Let g be an L^1 function such that $\int_{-\infty}^{\infty} g(y)\, dy = 1$, and let $\alpha = \int_{-\infty}^{0} g(y)\, dy$ and $\beta = \int_0^\infty g(y)\, dy$. (Note that $\alpha + \beta = 1$ and that $\alpha = \beta = \frac{1}{2}$ if g is even.) Suppose that f is piecewise continuous on $\mathbb{R}$, and suppose either that f is bounded or that g vanishes outside a finite interval so that $f * g(x)$ is well-defined for all x. If g_ϵ is defined by (7.3), then*

$$\lim_{\epsilon \to 0} f * g_\epsilon(x) = \alpha f(x+) + \beta f(x-) \quad \text{for all } x.$$

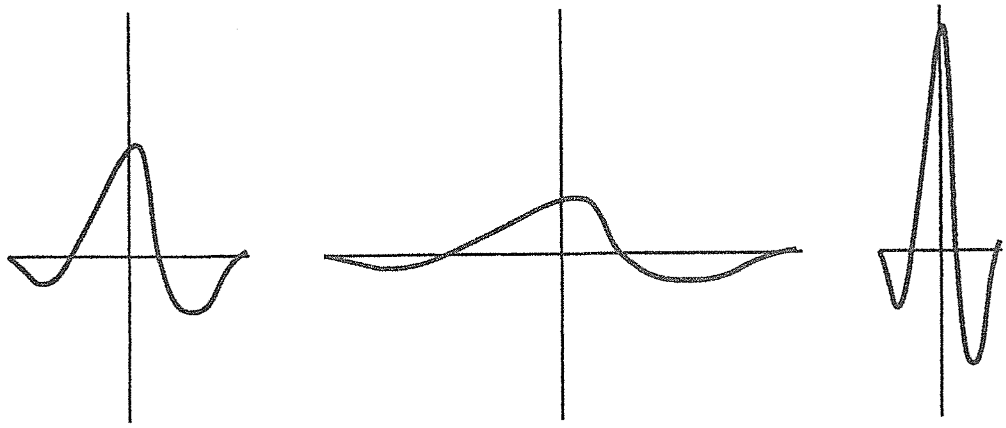


FIGURE 7.1. A function $g(x)$ (left) and its dilates $g_2(x) = \frac{1}{2}g(\frac{1}{2}x)$ (middle) and $g_{1/2}(x) = 2g(2x)$ (right).

In particular, if f is continuous at x, then

$$\lim_{\epsilon \to 0} f * g_\epsilon(x) = f(x). \tag{7.5}$$

Moreover, if f is continuous at every point in the bounded interval $[a, b]$, the convergence in (7.5) is uniform on $[a, b]$.

Proof: We have

$$f * g_\epsilon(x) - \alpha f(x+) - \beta f(x-) = \int_{-\infty}^{0} \Big[f(x-y) - f(x+)\Big] g_\epsilon(y)\, dy$$
$$+ \int_{0}^{\infty} \Big[f(x-y) - f(x-)\Big] g_\epsilon(y)\, dy,$$

so we wish to show that both integrals on the right can be made arbitrarily small by taking ϵ sufficiently small. The argument is the same for both of them, so we consider only the second one. Given $\delta > 0$, we can choose $c > 0$ small enough so that $|f(x-y) - f(x-)| < \delta$ when $0 < y < c$, and we break up the integral as $\int_0^c + \int_c^\infty$. By (7.4),

$$\left|\int_0^c \Big[f(x-y) - f(x-)\Big] g_\epsilon(y)\, dy\right| \le \delta \int_0^c |g_\epsilon(y)|\, dy = \delta \int_0^{c/\epsilon} |g(y)|\, dy$$
$$\le \delta \int_0^\infty |g(y)|\, dy,$$

and we can make this as small as we wish by choosing δ suitably. To estimate the integral from c to ∞, we use the assumption that either f is bounded (say $|f| \le M$) or g vanishes outside a finite interval (say $g(x) = 0$ for $|x| > R$). In the first case, by (7.4),

$$\left|\int_c^\infty \Big[f(x-y) - f(x-)\Big] g_\epsilon(y)\, dy\right| \le 2M \int_c^\infty |g_\epsilon(y)|\, dy = 2M \int_{c/\epsilon}^\infty |g(y)|\, dy,$$

which tends to zero along with ϵ. In the second case, $g_\epsilon(x) = 0$ for $|x| > \epsilon R$, and in particular $g_\epsilon(x) = 0$ for $x > c$ if $\epsilon < c/R$, so the integral from c to ∞ actually vanishes for ϵ small.

Finally, if f is continuous on $[a, b]$, then f is uniformly continuous there, so the choice of c in the preceding argument can be made independent of x for $x \in [a, b]$. It follows easily that the convergence of $f * g_\epsilon(x)$ to $f(x)$ is uniform on $[a, b]$. ∎

There are several variants of Theorem 7.3, which say that $f * g_\epsilon \to f$ in some sense or other as $\epsilon \to 0$ under suitable hypotheses on f and g. We shall content ourselves with stating a result for norm convergence of L^2 functions.

Theorem 7.4. *Suppose $g \in L^1$ is bounded and satisfies $\int g(y)\,dy = 1$. If $f \in L^2$, then $f * g(x)$ is well-defined for all x, and if g_ϵ is defined as in (7.3), $f * g_\epsilon$ converges to f in norm as $\epsilon \to 0$.*

The proof of this result is not really difficult, but it involves some approximation arguments that are a bit beyond the level of the present discussion. See Folland [25], Theorem 8.14, or Wheeden-Zygmund [56], Theorem 9.6.

The family $\{g_\epsilon\}$ in Theorems 7.3 and 7.4 is called an **approximate identity**, since the operation of convolution with g_ϵ tends to the identity operator as $\epsilon \to 0$. See Figure 7.2.

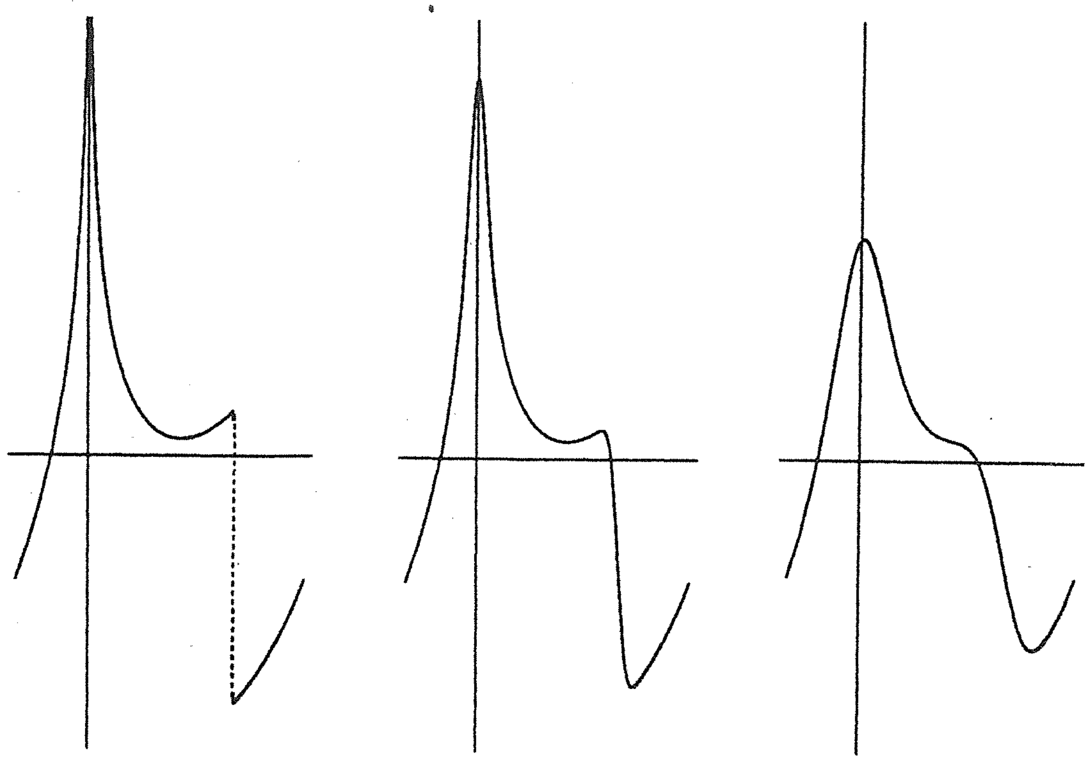


FIGURE 7.2. A function f with an infinite singularity and a jump discontinuity (left), $f * G_{0.1}$ (middle), and $f * G_{0.3}$ (right), where G is the Gaussian (7.6).

One of the functions g that is most often used in this context is the **Gaussian**

$$G(y) = \pi^{-1/2}e^{-y^2}. \tag{7.6}$$

It satisfies $\int G(y)\,dy = 1$ because

$$\int_{-\infty}^{\infty} e^{-y^2}dy = 2\int_0^{\infty} e^{-y^2}dy = \int_0^{\infty} e^{-t}t^{-1/2}dt = \Gamma(\tfrac{1}{2}) = \pi^{1/2}. \tag{7.7}$$

G is even, so that when it is used as the g in Theorem 7.3 we have $\alpha = \beta = \frac{1}{2}$. G and its dilated versions G_ϵ have the property that all their derivatives are bounded integrable functions. Indeed, it is easily established by induction that $G^{(k)}(y) = P_k(y)e^{-y^2}$ where P_k is a polynomial of degree k, and it follows that $|G^{(k)}(y)| \le C_k e^{-|y|}$, with similar estimates (involving some powers of ϵ) for G_ϵ. Hence we can apply Theorems 7.3 and 7.4: If f is (say) bounded and piecewise continuous, then $f * G_\epsilon$ is of class $C^{(\infty)}$, and it approximates f when ϵ is small. These convolutions may be regarded as "smeared out" or "smoothed out" versions of f. What we have developed here is a method of approximating general functions by smooth ones, a useful technical tool in many situations. In particular, it yields a proof of the following fundamental result.

The Weierstrass Approximation Theorem. *If f is a continuous function on $[a, b]$ $(-\infty < a < b < \infty)$, then f is the uniform limit of polynomials on $[a, b]$. That is, for any $\delta > 0$ there is a polynomial P such that*

$$\sup_{a\le x\le b} |f(x) - P(x)| < \delta.$$

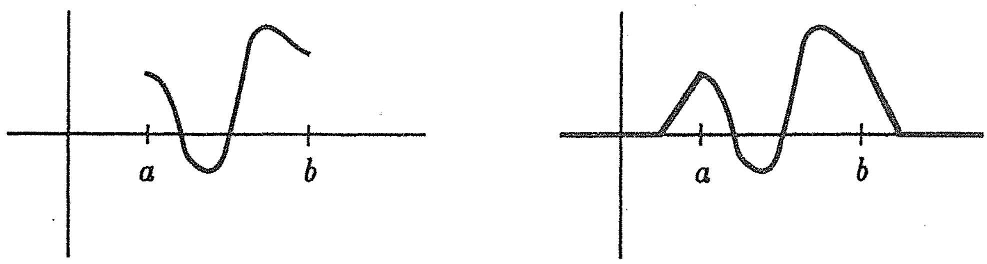


FIGURE 7.3. A continuous function f on $[a, b]$ (left) and a continuous extension of f to $\mathbb{R}$ (right).

Proof: Extend f to be a continuous function on the whole real line that vanishes outside the interval $[a-1, b+1]$; see Figure 7.3. By Theorem 7.3, $f * G_\epsilon \to f$ uniformly on $[a, b]$ where G is given by (7.6). Thus, given $\delta > 0$, if ϵ is sufficiently small we have

$$\sup_{a\le x\le b}\left| f(x) - \frac{1}{\epsilon\sqrt{\pi}}\int_{a-1}^{b+1} e^{-(x-y)^2/\epsilon^2} f(y)\,dy\right| < \frac{\delta}{2}.$$

As x ranges over $[a,b]$ and y ranges over $[a-1,\ b+1]$, $(x-y)/\epsilon$ ranges over the bounded set $[c,d]$ where $c=(a-b-1)/\epsilon$ and $d=(b-a+1)/\epsilon$, and the Taylor series $\sum_0^\infty(-1)^n t^{2n}/n!$ for e^{-t^2} converges uniformly on this set. It follows easily that we can replace $e^{-(x-y)^2/\epsilon^2}$ in the above integral by a suitable Taylor polynomial without changing the integral by more than $\frac{1}{2}\delta$. In other words, if N is sufficiently large,

$$\sup_{a\le x\le b}|f(x)-P(x)|<\delta$$

where

$$P(x)=\frac{1}{\epsilon\sqrt{\pi}}\sum_0^N\int_{a-1}^{b+1}\frac{(-1)^n(x-y)^{2n}}{\epsilon^{2n}n!}f(y)\,dy.$$

But $P(x)$ is a polynomial of degree $2N$, as one can see by expanding $(x-y)^{2n}$ by the binomial theorem:

$$P(x)=\sum_{n=0}^{N}\sum_{k=0}^{2n}c_{kn}x^k,\quad c_{kn}=\frac{(-1)^{k-n}(2n)!}{\epsilon^{2n+1}n!k!(2n-k)!\sqrt{\pi}}\int_{a-1}^{b+1}y^{2n-k}f(y)\,dy. \quad ∎$$

The Gaussian is not the only commonly used approximate identity. Another one is given by

$$H(y)=\frac{1}{\pi(1+y^2)},$$

which, as we shall see, arises in the solution of the Dirichlet problem for a half-plane. It shares with G the properties of being even and having derivatives of all orders that are bounded integrable functions, so it also provides smooth approximations to general bounded functions. Another approximate identity with these properties, and an extra one that makes it particularly useful in some situations, is given by

$$K(y)=\begin{cases}C^{-1}e^{-1/(1-y^2)} & \text{for } |y|<1,\\ 0 & \text{for } |y|\ge 1,\end{cases}\qquad C=\int_{-1}^{1}e^{-1/(1-y^2)}dy. \tag{7.8}$$

K possesses derivatives of all orders, even at $y=\pm1$ (because $e^{-1/(1-y^2)}$ vanishes to infinite order as y approaches 1 from the left or -1 from the right), and it vanishes outside the bounded set $|y|\le 1$. Hence the convolutions $f*K_\epsilon$ are well-defined for any piecewise continuous f, bounded or not, and they provide smooth approximations to all such f. Some other applications of K are given in Exercises 7 and 8.

EXERCISES

1. Which of the following functions are in L^1? in L^2?

   a. $\dfrac{\sin x}{|x|^{3/2}}$ b. $(1+x^2)^{-1/2}$ c. $\dfrac{1}{x^2-1}$ d. $\dfrac{1-\cos x}{x^2}$

2. Let $f(x) = |x|^{-p}$ where $\frac{1}{2} < p < 1$. Show that f is in neither L^1 nor L^2, but that f can be expressed as the sum of an L^1 function and an L^2 function.
3. Let $f(x) = 1$ if $-1 < x < 1$, $f(x) = 0$ otherwise.
   a. Compute $f * f$ and $f * f * f$.
   b. Let $f_\epsilon(x) = \epsilon^{-1} f(\epsilon^{-1}x)$ as in (7.3) and let $g(x) = x^3 - x$. Compute $f_\epsilon * g$ and check directly that $f_\epsilon * g \to 2g$ as $\epsilon \to 0$. (Note that $2 = \int f(x)\,dx$.)
4. Let $f(x) = e^{-x^2}$ and $g(x) = e^{-2x^2}$. Compute $f * g$. (Hint: Complete the square in the exponent and use the fact that $\int e^{-x^2}dx = \sqrt{\pi}$.)
5. For $t > 0$, let $f_t(x) = (4\pi t)^{-1/2} e^{-x^2/4t}$. Show that $f_t * f_s = f_{t+s}$. (Hint: First do Exercise 4 as a warmup.)
6. For $t > 0$, let $f_t(x) = x^{t-1}/\Gamma(t)$ for $x > 0$ and $f_t(x) = 0$ for $x \le 0$. Show that $f_t * f_s = f_{t+s}$. (Hint: The integral defining f_{t+s} can be reduced to the integral for the beta function.)
7. Show that for any $\delta > 0$ there is a function ϕ on $\mathbb{R}$ with the following properties: (i) ϕ is of class $C^{(\infty)}$, (ii) $0 \le \phi(x) \le 1$ for all x, (iii) $\phi(x) = 1$ when $0 \le x \le 1$, (iv) $\phi(x) = 0$ when $x < -\delta$ or $x > 1 + \delta$. (Hint: Define f by $f(x) = 1$ if $-\frac{1}{2}\delta \le x \le 1 + \frac{1}{2}\delta$, $f(x) = 0$ otherwise. Show that $f * K_\epsilon$ does the job if K is as in (7.8) and $\epsilon < \frac{1}{2}\delta$.)
8. Show that for any $f \in L^2$ and any $\delta > 0$, there is a function g such that (i) g is of class $C^{(\infty)}$, (ii) g vanishes outside a finite interval, and (iii) $\|f - g\| < \delta$. Proceed by the following steps.
   a. Let $F(x) = f(x)$ if $|x| < N$, $F(x) = 0$ otherwise. Show that $\|F - f\| < \frac{1}{2}\delta$ if N is sufficiently large.
   b. Show that $g = F * K_\epsilon$ does the job if K is as in (7.8) and ϵ is sufficiently small.

7.2 The Fourier transform

If f is an integrable function on $\mathbb{R}$, its **Fourier transform** is the function $\widehat{f}$ on $\mathbb{R}$ defined by

$$\widehat{f}(\xi) = \int e^{-i\xi x} f(x)\,dx.$$

We shall sometimes write $f^\wedge$ instead of $\widehat{f}$, particularly when the label f is replaced by a more complicated expression. We shall also occasionally write

$$\mathcal{F}\big[f(x)\big] = \widehat{f}(\xi)$$

for the Fourier transform of f. (This involves an ungrammatical use of the symbols x and ξ but is sometimes the clearest way of expressing things.)

Since $e^{-i\xi x}$ has absolute value 1, the integral converges absolutely for all ξ and defines a bounded function of ξ:

$$|\widehat{f}(\xi)| \le \int |f(x)|\,dx. \tag{7.9}$$

Moreover, since $|e^{-i\eta x}f(x) - e^{-i\xi x}f(x)| \le 2|f(x)|$, the dominated convergence theorem implies that $\widehat{f}(\eta) - \widehat{f}(\xi) \to 0$ when $\eta \to \xi$, that is, $\widehat{f}$ is continuous.

The following theorem summarizes some of the other basic properties of the Fourier transform.

Theorem 7.5. *Suppose $f \in L^1$.*
(a) For any $a \in \mathbb{R}$,

$$\mathcal{F}\left[f(x-a)\right] = e^{-ia\xi}\widehat{f}(\xi) \quad and \quad \mathcal{F}\left[e^{iax}f(x)\right] = \widehat{f}(\xi - a).$$

(b) If $\delta > 0$ and $f_\delta(x) = \delta^{-1}f(x/\delta)$ as in (7.3), then

$$[f_\delta]^\frown(\xi) = \widehat{f}(\delta\xi) \quad and \quad \mathcal{F}\left[f(\delta x)\right] = [\widehat{f}]_\delta(\xi).$$

(c) If f is continuous and piecewise smooth and $f' \in L^1$, then

$$[f']^\frown(\xi) = i\xi\widehat{f}(\xi).$$

On the other hand, if $xf(x)$ is integrable, then

$$\mathcal{F}\left[xf(x)\right] = i[\widehat{f}]'(\xi).$$

(d) If also $g \in L^1$, then

$$(f * g)^\frown = \widehat{f}\widehat{g}.$$

Proof: For the first equation of (a), we have

$$\mathcal{F}\left[f(x-a)\right] = \int e^{-i\xi x}f(x-a)\,dx = \int e^{-i\xi x - i\xi a}f(x)\,dx = e^{-ia\xi}\widehat{f}(\xi).$$

The other equations of (a) and (b) are equally easy to prove; we leave them as exercises for the reader. As for (c), observe that since $f' \in L^1$, the limit

$$\lim_{x\to+\infty} f(x) = f(0) + \int_0^\infty f'(x)\,dx$$

exists, and since $f \in L^1$ this limit must be zero. Likewise, $\lim_{x\to-\infty} f(x) = 0$. Hence we can integrate by parts, and the boundary terms vanish:

$$[f']^\frown(\xi) = \int e^{-i\xi x}f'(x)\,dx = -\int(-i\xi)e^{-i\xi x}f(x)\,dx = i\xi\widehat{f}(\xi).$$

On the other hand, if $xf(x)$ is integrable, since $xe^{-i\xi x} = i(d/d\xi)e^{-i\xi}$ we have

$$\mathcal{F}\left[xf(x)\right] = \int e^{-i\xi x}xf(x)\,dx = i\frac{d}{d\xi}\int e^{-i\xi x}f(x)\,dx = i[\widehat{f}]'(\xi).$$

Finally, for (d),

$$\begin{aligned}(f*g)\hat{}(\xi) &= \iint e^{-i\xi x} f(x-y)g(y)\,dy\,dx \\ &= \iint e^{-i\xi(x-y)} f(x-y)e^{-i\xi y} g(y)\,dx\,dy \\ &= \iint e^{-i\xi z} f(z)e^{-i\xi y} g(y)\,dz\,dy \qquad (z=x-y) \\ &= \hat{f}(\xi)\hat{g}(\xi).\end{aligned}$$

∎

Parts (a), (b), and (c) exhibit a remarkable set of correspondences between functions and their Fourier transforms. In essence: Translating a function corresponds to multiplying its Fourier transform by an exponential and vice versa; dilating a function by the factor δ corresponds to dilating its Fourier transform by the factor $1/\delta$ and vice versa; differentiating a function corresponds to multiplying its Fourier transform by the coordinate variable and vice versa. (Of course, this formulation is a bit imprecise; there are factors of -1, i, and δ to be sorted out.) This symmetry between f and $\hat{f}$ extends also to part (d): It will follow from (d) and the Fourier inversion formula below that

$$\hat{f} * \hat{g} = 2\pi (fg)\hat{}. \tag{7.10}$$

Before developing the theory further, let us compute three basic examples of Fourier transforms.

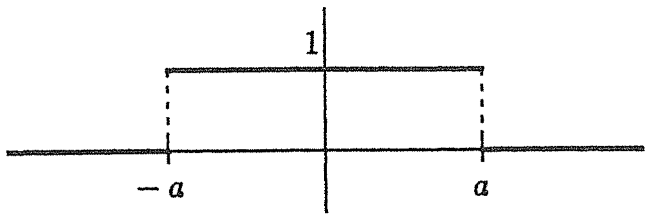


FIGURE 7.4. Graph of the function χ_a.

Example 1. Let χ_a be the function depicted in Figure 7.4:

$$\chi_a(x) = \begin{cases} 1 & \text{if } |x| < a, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\hat{\chi}_a(\xi) = \int_{-a}^{a} e^{-i\xi x}\,dx = \frac{e^{-ia\xi} - e^{ia\xi}}{-i\xi} = 2\frac{\sin a\xi}{\xi}. \tag{7.11}$$

Example 2. Let $f(x) = e^{-ax^2/2}$ where $a > 0$. We observe that f satisfies the differential equation $f'(x) + axf(x) = 0$. If we apply the Fourier transform to this equation, by Theorem 7.5(c) we obtain $i\xi\widehat{f}(\xi) + ia[\widehat{f}]'(\xi) = 0$, or $[\widehat{f}]'(\xi) + a^{-1}\xi\widehat{f}(\xi) = 0$. This differential equation for $\widehat{f}$ is easily solved:

$$\frac{[\widehat{f}]'(\xi)}{\widehat{f}(\xi)} = -\frac{\xi}{a} \implies \log\widehat{f}(\xi) = -\frac{\xi^2}{2a} + \log C \implies \widehat{f}(\xi) = Ce^{-\xi^2/2a}.$$

To evaluate the constant C, we set $\xi = 0$ and use (7.7):

$$C = \widehat{f}(0) = \int f(x)\,dx = \int e^{-ax^2/2}\,dx = \sqrt{\frac{2}{a}}\int e^{-y^2}\,dy = \sqrt{\frac{2\pi}{a}}.$$

Therefore,

$$\mathscr{F}\left[e^{-ax^2/2}\right] = \sqrt{\frac{2\pi}{a}}e^{-\xi^2/2a}. \tag{7.12}$$

(A neat derivation of this result using contour integrals is sketched in Exercise 1.)

Example 3. Let $f(x) = (x^2 + a^2)^{-1}$ where $a > 0$. We shall calculate $\widehat{f}$ here by contour integration; another derivation that uses the Fourier inversion formula but no complex variable theory is sketched in Exercise 2. If $\xi < 0$, $e^{-i\xi z}$ is a bounded analytic function of z in the upper half-plane, so by applying the residue theorem on the contour in Figure 7.5 and letting $N \to \infty$ we obtain

$$\widehat{f}(\xi) = \int \frac{e^{-i\xi x}}{x^2 + a^2}\,dx = 2\pi i\,\mathrm{Res}_{z=i}\frac{e^{-i\xi z}}{(z^2 + a^2)} = 2\pi i\frac{e^{a\xi}}{2ia} = \frac{\pi}{a}e^{a\xi} \qquad (\xi < 0).$$

Similarly, if $\xi > 0$ we can integrate around the lower half-plane to obtain $\widehat{f}(\xi) = (\pi/a)e^{-a\xi}$. Of course, for $\xi = 0$, $\widehat{f}(0) = \int f(x)\,dx = \pi/a$. Conclusion:

$$\mathscr{F}\left[(x^2 + a^2)^{-1}\right] = \frac{\pi}{a}e^{-a|\xi|}. \tag{7.13}$$

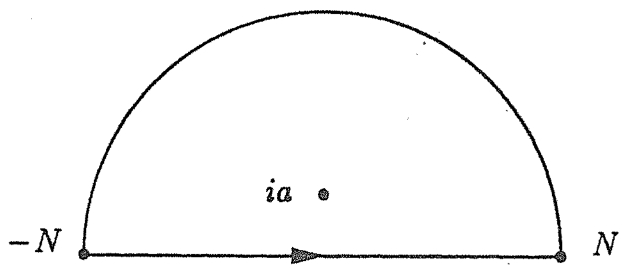


FIGURE 7.5. The contour for Example 3.

The equation $[f']\widehat{\ }(\xi) = i\xi\widehat{f}(\xi)$ of Theorem 7.5(c) is the analogue of Theorem 2.2 in §2.3 for Fourier series. The moral here, as in the theory of Fourier series, is that the smoother f is, the faster $\widehat{f}$ decays at infinity, and vice versa. The examples worked out above illustrate this principle. The function of Example 1 vanishes outside a finite interval but is not continuous; its Fourier transform is analytic but decays only like $1/\xi$ at infinity. The function of Example 3 is smooth but decays slowly; its Fourier transform decays rapidly but is not differentiable at $\xi = 0$. The function of Example 2 has both smoothness and decay, and is essentially its own Fourier transform.

One other basic property of Fourier transforms of L^1 functions should be mentioned here. We observed earlier that if $f \in L^1$, then $\widehat{f}$ is a bounded, continuous function on $\mathbb{R}$; but something more is true.

The Riemann-Lebesgue Lemma. *If $f \in L^1$, then $\widehat{f}(\xi) \to 0$ as $\xi \to \pm\infty$.*

Proof: First suppose that f is a step function, that is, $f(x) = \sum_1^k c_j\phi_j(x)$ where each ϕ_j is a function that equals 1 on some bounded interval $|x - x_j| < a_j$ and equals 0 elsewhere. By (7.11) and Theorem 7.5(a), $\widehat{\phi}_j(\xi) = 2\xi^{-1}e^{-ix_j\xi}\sin a_j\xi$, which vanishes at infinity. Hence, so does $\widehat{f}$.

For the general case, if $f \in L^1$ one can find a sequence $\{f_n\}$ of step functions such that $\int |f_n(x) - f(x)|\,dx \to 0$. (When f is Riemann integrable, this assertion is essentially a restatement of the fact that the integral of f is the limit of Riemann sums. It is true also for Lebesgue integrable functions, but the proof naturally requires some results from Lebesgue integration theory. See Folland [25], Theorems 2.26 and 2.41.) But then by (7.9),

$$\sup_\xi |\widehat{f}_n(\xi) - \widehat{f}(\xi)| \le \int |f_n(x) - f(x)|\,dx \to 0 \quad \text{as } n \to \infty,$$

that is, $\widehat{f}_n \to \widehat{f}$ uniformly. Since each $\widehat{f}_n$ vanishes at infinity, it follows easily that $\widehat{f}$ does too. ∎

The Fourier inversion theorem

We now turn to the Fourier inversion formula, that is, the procedure for recovering f from $\widehat{f}$. The heuristic arguments in the introduction to this chapter led us to the formula

$$f(x) = \frac{1}{2\pi}\int e^{i\xi x}\widehat{f}(\xi)\,d\xi. \tag{7.14}$$

(Note that this is the same as the formula that gives $\widehat{f}$ in terms of f, except for the plus sign in the exponent and the factor of 2π. This accounts for the symmetry between f and $\widehat{f}$ in Theorem 7.5.) Our task is to investigate the validity of (7.14). Like the question of whether the Fourier series of a periodic function f converges to f, this is not entirely straightforward.

The first difficulty is that $\widehat{f}$ may not be in L^1, as (7.11) shows, and in this case the integral in (7.14) is not absolutely convergent. Even if it is, one cannot establish (7.14) simply by substituting in the defining formula for $\widehat{f}$,

$$\int e^{i\xi x}\widehat{f}(\xi)\,d\xi = \iint e^{i\xi(x-y)}f(y)\,dy\,d\xi,$$

and interchanging the order of integration, because the integral $\int e^{i\xi(x-y)}d\xi$ is divergent. The simplest remedy for both these problems is to multiply $\widehat{f}$ by a "cutoff function" to make the integrals converge and then to pass to the limit as the cutoff is removed.

One convenient cutoff function is $e^{-\epsilon^2\xi^2/2}$: For any fixed $\epsilon > 0$ it decreases rapidly as $\xi \to \pm\infty$, and to remove it we simply let $\epsilon \to 0$. Accordingly, instead of (7.14), for $f \in L^1$ we consider

$$\frac{1}{2\pi}\int e^{i\xi x}e^{-\epsilon^2\xi^2/2}\widehat{f}(\xi)\,d\xi = \frac{1}{2\pi}\iint e^{i\xi(x-y)}e^{-\epsilon^2\xi^2/2}f(y)\,dy\,d\xi.$$

Now the double integral is absolutely convergent and it is permissible to interchange the order of integration. The ξ-integral is evaluated by (7.12):

$$\int e^{i\xi(x-y)}e^{-\epsilon^2\xi^2/2}d\xi = \mathscr{F}\left[e^{-\epsilon^2\xi^2/2}\right](y-x) = \frac{\sqrt{2\pi}}{\epsilon}e^{-(x-y)^2/2\epsilon^2}.$$

In other words,

$$\frac{1}{2\pi}\int e^{i\xi x}e^{-\epsilon^2\xi^2/2}\widehat{f}(\xi)\,d\xi = \frac{1}{\epsilon\sqrt{2\pi}}\int f(y)e^{-(x-y)^2/2\epsilon^2}dy = f * \phi_\epsilon(x)$$

where

$$\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}, \qquad \phi_\epsilon(x) = \frac{1}{\epsilon}\phi\left(\frac{x}{\epsilon}\right) = \frac{1}{\epsilon\sqrt{2\pi}}e^{-x^2/2\epsilon^2}.$$

But this is precisely the situation of Theorem 7.3 and example (7.6) in §7.1 (with ϵ replaced by $\epsilon\sqrt{2}$), and we conclude that if f is piecewise continuous,

$$\lim_{\epsilon\to 0}\frac{1}{2\pi}\int e^{i\xi x}e^{-\epsilon^2\xi^2/2}\widehat{f}(\xi)\,d\xi = \tfrac{1}{2}\left[f(x-)+f(x+)\right]$$

for all x. We have arrived at our main result.

The Fourier Inversion Theorem. *Suppose f is integrable and piecewise continuous on $\mathbb{R}$, defined at its points of discontinuity so as to satisfy $f(x) = \frac{1}{2}\left[f(x-)+f(x+)\right]$ for all x. Then*

$$f(x) = \lim_{\epsilon\to 0}\frac{1}{2\pi}\int e^{i\xi x}e^{-\epsilon^2\xi^2/2}\widehat{f}(\xi)\,d\xi, \qquad x \in \mathbb{R}. \tag{7.15}$$

Moreover, if $\widehat{f} \in L^1$, then f is continuous and

$$f(x) = \frac{1}{2\pi}\int e^{i\xi x}\widehat{f}(\xi)\,d\xi, \qquad x \in \mathbb{R}. \tag{7.16}$$

Proof: The only thing left to prove is the last assertion. But

$$\left|e^{i\xi x}e^{-\epsilon^2|\xi|^2/2}\widehat{f}(\xi)\right| \le \left|\widehat{f}(\xi)\right|,$$

so if $\widehat{f} \in L^1$, we can apply the dominated convergence theorem to evaluate the limit in (7.15):

$$\frac{1}{2\pi}\int e^{i\xi x}\widehat{f}(\xi)\,d\xi = \lim_{\epsilon\to 0}\frac{1}{2\pi}\int e^{i\xi x}e^{-\epsilon^2\xi^2/2}\widehat{f}(\xi)\,d\xi = \tfrac{1}{2}\left[f(x-)+f(x+)\right].$$

The integral on the left is $1/2\pi$ times the Fourier transform of $\widehat{f}$ evaluated at $-x$, and we have already pointed out that Fourier transforms of integrable functions are continuous. Hence f is continuous and $\frac{1}{2}\left[f(x-)+f(x+)\right] = f(x)$. ∎

The inversion formula (7.16), or its variant (7.15), expresses a general function f as a continuous superposition of the exponential functions $e^{i\xi x}$. In this way it provides an analogue for nonperiodic functions of the Fourier series expansion of periodic functions.

Corollary 7.1. *If $\widehat{f} = \widehat{g}$, then $f = g$.*

Proof: If $\widehat{f} = \widehat{g}$, then $(f-g)\widehat{\ } = 0$, so $f - g = 0$ by (7.16). ∎

If ϕ is the Fourier transform of $f \in L^1$, we say that f is the **inverse Fourier transform** of ϕ and write $f = \mathscr{F}^{-1}\phi$. The operation $\mathscr{F}^{-1}$ is well-defined by Corollary 7.1.

Remark. Functions f such that f and $\widehat{f}$ are both in L^1 exist in great abundance; one needs only a little smoothness of f to ensure the necessary decay of $\widehat{f}$ at infinity. For example, if f is twice differentiable and f' and f'' are also integrable, then $(f'')\widehat{\ }(\xi) = -\xi^2\widehat{f}(\xi)$ is bounded, so $|\widehat{f}(\xi)| \le C/(1+\xi^2)$, whence $\widehat{f} \in L^1$. (See also Exercise 7.) Such functions have the property that f and $\widehat{f}$ are bounded and continuous as well as integrable, and hence f and $\widehat{f}$ are also in L^2.

A number of variations on the Fourier inversion theorem are possible. For one thing, a version of (7.15) is true for functions $f \in L^1$ that are not piecewise continuous; namely, if $f \in L^1$, we have

$$f(x) = \lim_{\epsilon\to 0}\frac{1}{2\pi}\int e^{i\xi x}e^{-\epsilon^2\xi^2/2}\widehat{f}(\xi)\,d\xi$$

for "almost every" $x \in \mathbf{R}$, in the sense of Lebesgue measure. For another, one can replace the cutoff function $e^{-\epsilon^2\xi^2/2}$ in (7.15) by any of a large number of other functions with similar properties. (See Folland [25], Theorem 8.31; also Exercise 5.)

On the more naive level, one can ask whether the integral in (7.14) can be interpreted simply as a (principal value) improper integral, that is, whether

$$f(x) = \lim_{r\to\infty}\frac{1}{2\pi}\int_{-r}^{r} e^{i\xi x}\widehat{f}(\xi)\,d\xi.$$

This amounts to using the cutoff function that equals 1 on $[-r, r]$ and 0 elsewhere, and letting $r \to \infty$; it is the obvious analogue of evaluating a Fourier series as the limit of its symmetric partial sums as we did in §2.2. Just as in that case, piecewise continuity of f does not suffice, but piecewise smoothness does.

Theorem 7.6. *If f is integrable and piecewise smooth on $\mathbb{R}$, then*

$$\lim_{r\to\infty} \frac{1}{2\pi}\int_{-r}^{r} e^{i\xi x}\widehat{f}(\xi)\,d\xi = \tfrac{1}{2}\big[f(x-)+f(x+)\big] \tag{7.17}$$

for every $x \in \mathbb{R}$.

Proof: We have

$$\int_{-r}^{r} e^{i\xi(x-y)}d\xi = \frac{e^{ir(x-y)} - e^{-ir(x-y)}}{i(x-y)} = \frac{2\sin r(x-y)}{x-y},$$

so

$$\begin{aligned}\frac{1}{2\pi}\int_{-r}^{r} e^{i\xi x}\widehat{f}(\xi)\,d\xi &= \frac{1}{2\pi}\int_{-r}^{r}\int e^{i\xi(x-y)}f(y)\,dy\,d\xi\\ &= \frac{1}{\pi}\int \frac{\sin r(x-y)}{x-y}f(y)\,dy = \frac{1}{\pi}\int \frac{\sin ry}{y}f(x-y)\,dy.\end{aligned}$$

This has the form $f * g_\epsilon(x)$ where $\epsilon = 1/r$ and $g(x) = (\sin x)/x$, just as in the arguments leading to (7.15). The trouble is that $(\sin x)/x$ is not in L^1, so Theorem 7.3 is not applicable. (See Exercise 6.) However, it is well known that

$$\int_0^\infty \frac{\sin ry}{y}\,dy = \int_{-\infty}^0 \frac{\sin ry}{y}\,dy = \frac{\pi}{2},$$

where the integrals are conditionally convergent. (See Boas [8], §9D.) Hence, we can write

$$\begin{aligned}&\frac{1}{2\pi}\int_{-r}^{r} e^{i\xi x}\widehat{f}(\xi)\,d\xi - \tfrac{1}{2}\big[f(x-)+f(x+)\big]\\ &\quad= \frac{1}{\pi}\int_{-\infty}^{0}\frac{\sin ry}{y}\big[f(x-y)-f(x+)\big]\,dy + \frac{1}{\pi}\int_0^\infty \frac{\sin ry}{y}\big[f(x-y)-f(x-)\big]\,dy,\end{aligned}$$

and it suffices to show that both integrals on the right tend to zero as $r \to \infty$.

Consider the integral over $(0,\infty)$; the argument for the other one is much the same. For any $K > 0$ we can write it as

$$\int_0^K \frac{\sin ry}{y}\big[f(x-y)-f(x-)\big] + \int_K^\infty \frac{\sin ry}{y}\big[f(x-y)-f(x-)\big]\,dy. \tag{7.18}$$

If $K \geq 1$ we have

$$\left|\int_K^\infty \frac{\sin ry}{y}f(x-y)\,dy\right| \leq \int_K^\infty |f(x-y)|\,dy$$

and

$$\int_K^\infty \frac{\sin ry}{y} f(x-)\,dy = f(x-)\int_{rK}^\infty \frac{\sin z}{z}\,dz.$$

These are the tail ends of convergent integrals, so they tend to zero as $K \to \infty$, no matter what r is (as long as, say, $r \geq 1$). Hence we can make the integral over (K, ∞) in (7.18) as small as we wish by taking K sufficiently large.

On the other hand, the integral over $(0, K)$ in (7.18) equals

$$\int \frac{e^{iry} - e^{-iry}}{2i} g(y)\,dy = \frac{1}{2i}\left[\widehat{g}(r) - \widehat{g}(-r)\right]$$

where

$$g(y) = \frac{f(x-y) - f(x-)}{y} \text{ if } 0 < y < K, \qquad g(y) = 0 \text{ otherwise.}$$

Since f is piecewise smooth, g is piecewise smooth except perhaps near $y = 0$, and $g(y)$ approaches the finite limit $f'(x-)$ as y decreases to 0. Hence g is bounded on $[0, K]$ and thus is integrable on $\mathbb{R}$. But then $\widehat{g}(r)$ and $\widehat{g}(-r)$ tend to zero as $r \to \infty$ by the Riemann-Lebesgue lemma, so we are done. ∎

The Fourier transform on L^2

We have developed the Fourier transform in the setting of the space L^1, but our experience with Fourier series suggests that the space L^2 should also play a significant role. This is indeed the case. There is an initial difficulty to be overcome, in that the integral $\int e^{-i\xi x} f(x)\,dx$ may not converge if f is in L^2 but not in L^1, but there is a way around this problem. The key observation is that the analogue of Parseval's formula holds for the Fourier transform. Namely, suppose that f and g are L^1 functions such that $\widehat{f}$ and $\widehat{g}$ are in L^1. Then f, g, $\widehat{f}$, and $\widehat{g}$ are also in L^2 (cf. the remark following the Fourier inversion theorem), and by (7.16) we have

$$\begin{aligned} 2\pi\langle f, g\rangle &= 2\pi \int f(x)\overline{g(x)}\,dx = \iint f(x)\overline{e^{i\xi x}\widehat{g}(\xi)}\,d\xi\,dx \\ &= \iint f(x)e^{-i\xi x}\overline{\widehat{g}(\xi)}\,dx\,d\xi = \int \widehat{f}(\xi)\overline{\widehat{g}(\xi)}\,d\xi = \langle \widehat{f}, \widehat{g}\rangle. \end{aligned}$$

In other words, the Fourier transform preserves inner products up to a factor of 2π. In particular, taking $g = f$, we obtain

$$\|\widehat{f}\|^2 = 2\pi\|f\|^2,$$

which is the "Parseval formula" for the Fourier transform.

Now, if f is an arbitrary L^2 function, we can find a sequence $\{f_n\}$ such that f_n and $\widehat{f}_n$ are in L^1 and $f_n \to f$ in the L^2 norm. (This follows from Theorem

2.7 of §2.4, which we stated without any proof, but we now have the machinery to construct such sequences explicitly; see Exercise 8, §7.1.) Then

$$\|\widehat{f}_n - \widehat{f}_m\|^2 = 2\pi\|f_n - f_m\|^2 \to 0 \quad \text{as} \quad m, n \to \infty,$$

so $\{\widehat{f}_n\}$ is a Cauchy sequence in L^2. Since L^2 is complete, it has a limit, which is easily seen to depend only on f and not on the approximating sequence $\{f_n\}$. We *define* this limit to be $\widehat{f}$. In this way, the domain of the Fourier transform is extended to include all of L^2, and a simple limiting argument shows that this extended Fourier transform still preserves the norm and inner product up to a factor of 2π, and that it still satisfies the properties of Theorem 7.5. In short, we have the following result.

The Plancherel Theorem. *The Fourier transform, defined originally on $L^1 \cap L^2$, extends uniquely to a map from L^2 to itself that satisfies*

$$\langle \widehat{f}, \widehat{g} \rangle = 2\pi \langle f, g \rangle \quad \textit{and} \quad \|\widehat{f}\|^2 = 2\pi\|f\|^2 \quad \textit{for all } f, g \in L^2.$$

Moreover, the formulas of Theorem 7.5 still hold for L^2 functions.

If f is in L^2 but not in L^1, the integral $\int f(x)e^{-i\xi x}\,dx$ defining $\widehat{f}$ may not converge pointwise, but it may be interpreted by a limiting process like the one we used in the inversion formula (7.15). That is, if $f \in L^2$, as $\epsilon \to 0$ the functions g^ϵ defined by

$$g^\epsilon(\xi) = \int e^{-i\xi x} e^{-\epsilon^2 x^2/2} f(x)\,dx$$

converge in the L^2 norm, and pointwise almost everywhere, to $\widehat{f}$. Likewise, the functions f^ϵ defined by

$$f^\epsilon(x) = \frac{1}{2\pi} \int e^{i\xi x} e^{-\epsilon^2 \xi^2/2} \widehat{f}(\xi)\,d\xi$$

converge in the L^2 norm, and pointwise almost everywhere, to f.

The Fourier inversion theorem is also a useful device for computing Fourier transforms. Indeed, upon setting $\phi = \widehat{f}$ the inversion formula (7.16) can be restated as

$$\phi = \widehat{f} \implies f(x) = (2\pi)^{-1}\widehat{\phi}(-x).$$

(The original formula (7.16) is valid when ϕ and $\widehat{\phi}$ are in L^1, but in the present form it continues to hold for any $\phi \in L^2$.) But this means that if ϕ is the Fourier transform of a known function f, we can immediately write down the Fourier transform of ϕ by setting $\xi = -x$:

$$\phi = \widehat{f} \implies \widehat{\phi}(\xi) = 2\pi f(-\xi).$$

For example, from formula (7.11) we have

$$\mathscr{F}\left[\frac{\sin ax}{x}\right] = \begin{cases} \pi & \text{if } |\xi| < a, \\ 0 & \text{otherwise.} \end{cases}$$

TABLE 2. SOME BASIC FOURIER TRANSFORMS

Functions are listed on the left, their Fourier transforms on the right. a and c denote constants with $a > 0$ and $c \in \mathbb{R}$.

1.	$f(x)$	$\widehat{f}(\xi)$
2.	$f(x-c)$	$e^{-ic\xi}\widehat{f}(\xi)$
3.	$e^{icx}f(x)$	$\widehat{f}(\xi - c)$
4.	$f(ax)$	$a^{-1}\widehat{f}(a^{-1}\xi)$
5.	$f'(x)$	$i\xi\widehat{f}(\xi)$
6.	$xf(x)$	$i(\widehat{f})'(\xi)$
7.	$(f * g)(x)$	$\widehat{f}(\xi)\widehat{g}(\xi)$
8.	$f(x)g(x)$	$(2\pi)^{-1}(\widehat{f} * \widehat{g})(\xi)$
9.	$e^{-ax^2/2}$	$\sqrt{2\pi/a}\, e^{-\xi^2/2a}$
10.	$(x^2+a^2)^{-1}$	$(\pi/a)e^{-a\lvert\xi\rvert}$
11.	$e^{-a\lvert x\rvert}$	$2a(\xi^2+a^2)^{-1}$
12.	$\chi_a(x) = \begin{cases} 1 & (\lvert x\rvert < a) \\ 0 & (\lvert x\rvert > a) \end{cases}$	$2\xi^{-1}\sin a\xi$
13.	$x^{-1}\sin ax$	$\pi\chi_a(\xi) = \begin{cases} \pi & (\lvert\xi\rvert < a) \\ 0 & (\lvert\xi\rvert > a) \end{cases}$

Here $x^{-1}\sin ax$ is a function that is in L^2 but not in L^1; the calculation of its Fourier transform directly from the definition is a somewhat tricky business.

Table 2 contains a brief list of basic Fourier transform formulas that we have derived in this section. All of them will be used repeatedly in what follows. Much more extensive tables of Fourier transforms are available — for example, Erdélyi et al. [22]. (Most of the entries in [22] are in the form of Fourier sine or cosine transforms; see §7.4, especially the concluding remarks.)

One final remark: The definition of the Fourier transform that we have adopted here is not universally accepted. Two other frequently used definitions are

$$\widetilde{f}(\xi) = \frac{1}{\sqrt{2\pi}}\int e^{-i\xi x} f(x)\,dx, \qquad \check{f}(\xi) = \int e^{-2\pi i\xi x} f(x)\,dx,$$

for which the inversion formulas are

$$f(x) = \frac{1}{\sqrt{2\pi}}\int e^{i\xi x}\widetilde{f}(\xi)\,d\xi, \qquad f(x) = \int e^{2\pi i\xi x}\check{f}(\xi)\,d\xi.$$

Some people also omit the minus sign in the exponent in the formulas defining $\widehat{f}$, $\widetilde{f}$, and $\check{f}$; it then reappears in the exponent in the inversion formula. $\widetilde{f}$ has the advantage of getting rid of the 2π in the Plancherel theorem, $\|\widetilde{f}\|^2 = \|f\|^2$, but the disadvantage of introducing one in the convolution formula, $(f * g)^{\widehat{}} = \sqrt{2\pi}\widehat{f}\widehat{g}$. $\check{f}$ obviates the 2π's in both these formulas, $\|\check{f}\|^2 = \|f\|^2$ and $(f * g)^{\check{}} = \check{f}\check{g}$, but introduces them in the formula for derivatives, $(f')^{\check{}}(\xi) = 2\pi i\xi\check{f}(\xi)$. In short, one can choose to put the 2π's where one finds them least annoying, but one cannot get rid of them entirely.

EXERCISES

1. If $f(x) = e^{-ax^2/2}$ with $a > 0$, then $\widehat{f}(\xi) = \int e^{-i\xi x - ax^2/2}dx$. Derive formula (7.12) by completing the square in the exponent, using Cauchy's theorem to shift the path of integration from the real axis ($\operatorname{Im} x = 0$) to the horizontal line $\operatorname{Im} x = -\xi/a$, and finally using (7.7).
2. Show directly that $\mathcal{F}\left[e^{-a|x|}\right] = 2a(\xi^2 + a^2)^{-1}$ and hence derive (7.13) from the Fourier inversion formula.
3. Complete the proof of Theorem 7.5(a, b).
4. Let f be as in Exercise 3, §7.1. Compute $\widehat{f}$ and $(f * f)^{\widehat{}}$ from the formulas in that exercise and verify that $(f * f)^{\widehat{}} = (\widehat{f})^2$.
5. Suppose $g \in L^1$, $\int g(x)\,dx = 1$, and $\widehat{g} \in L^1$.
   a. Show that $\widehat{g}(\delta\xi) \to 1$ as $\delta \to 0$ for all $\xi \in \mathbf{R}$.
   b. Show that for any continuous $f \in L^1$,
   $$\lim_{\delta\to 0}\frac{1}{2\pi}\int e^{i\xi x}\widehat{g}(\delta\xi)\widehat{f}(\xi)\,d\xi = f(x)$$
   for all x. What if f is only piecewise continuous? (Mimic the argument leading to (7.15), using the Fourier inversion theorem for g.)
6. Show that $\int_0^\infty x^{-1}|\sin x|\,dx = \infty$. (Hint: Show that $\int_{(n-1)\pi}^{n\pi} x^{-1}|\sin x|\,dx > 2/n$.)
7. Suppose that f is continuous and piecewise smooth, $f \in L^2$, and $f' \in L^2$. Show that $\widehat{f} \in L^1$. (Hint: First show that $\int(1+\xi^2)|\widehat{f}(\xi)|^2 d\xi$ is finite; then use the Cauchy-Schwarz inequality as in the proof of Theorem 2.3, §2.3.)
8. Given $a > 0$, let $f(x) = e^{-x}x^{a-1}$ for $x > 0$, $f(x) = 0$ for $x \le 0$. Show that $\widehat{f}(\xi) = \Gamma(a)(1 + i\xi)^{-a}$.
9. Use the residue theorem to show that
   $$\mathcal{F}\left[\frac{1}{x^4+1}\right] = \frac{\pi}{\sqrt{2}}e^{-|\xi|/\sqrt{2}}\left(\cos\frac{\xi}{\sqrt{2}} + \sin\frac{|\xi|}{\sqrt{2}}\right).$$
10. Let $f(x) = (\sinh ax)/(\sinh \pi x)$ where $0 < a < \pi$.
    a. Use the residue theorem to show that
    $$\widehat{f}(\xi) = 2i\sum_1^\infty(-1)^n e^{-n|\xi|}\sinh ina.$$
    b. Use the fact that $2\sinh ina = e^{ina} - e^{-ina}$ and sum the geometric series to show that
    $$\widehat{f}(\xi) = \frac{\sin a}{\cosh\xi + \cos a}.$$

11. Given $\nu > -\frac{1}{2}$, let $f(x) = (1 - x^2)^{\nu-(1/2)}$ if $|x| < 1$, $f(x) = 0$ if $|x| > 1$. Show that $\widehat{f}(\xi) = 2^\nu \pi^{1/2}\Gamma(\nu + \frac{1}{2})\xi^{-\nu} J_\nu(\xi)$. (Cf. Exercise 14, §5.2.)
12. For $a > 0$, let $f_a(x) = a/\left[\pi(x^2 + a^2)\right]$ and $g_a(x) = (\sin ax)/\pi x$. Use the Fourier transform to show that:
    a. $f_a * f_b = f_{a+b}$, b. $g_a * g_b = g_{\min(a,b)}$.
13. Use the Plancherel theorem to prove the indicated formulas. In all of them, a and b denote positive numbers.
    a. $\displaystyle\int \frac{\sin(at)\sin(bt)}{t^2}dt = \pi\min(a, b)$. (Cf. formula (7.11).)
    b. $\displaystyle\int \frac{t^2}{(t^2 + a^2)(t^2 + b^2)}dt = \frac{\pi}{a+b}$. (Cf. formula (7.13).)
    c. $\displaystyle\int (1 + it)^{-a}(1 - it)^{-b}dt = \frac{2^{2-a-b}\pi\Gamma(a + b - 1)}{\Gamma(a)\Gamma(b)}$. (Here $a, b > \frac{1}{2}$; use Exercise 8.)
14. Let h_n be the nth Hermite function as defined in (6.33). Show that $\widehat{h}_n(\xi) = \sqrt{2\pi}(-i)^n h_n(\xi)$. (Hint: Use induction on n. For h_0 the result is true by formula (7.12). Assuming the result for h_n, prove it for h_{n+1} by applying the Fourier transform to equation (6.40).) This shows that the Hermite functions are an orthogonal basis of eigenfunctions for the Fourier transform.
15. Let $l_n(x) = e^{-x/2}L_n^0(x)$ for $x > 0$, $l_n(x) = 0$ for $x < 0$, where L_n^0 denotes the Laguerre polynomial defined by (6.43), and let $\phi_n(\xi) = (2\pi)^{-1/2}\widehat{l}_n(\xi)$.
    a. Show that
    $$\phi_n(\xi) = \sqrt{\frac{2}{\pi}}\,\frac{(2i\xi - 1)^n}{(2i\xi + 1)^{n+1}}.$$
    (Hint: Plug the definition of L_n^0 into the formula defining $\widehat{l}_n$ and integrate by parts n times.)
    b. Deduce from part (a) and Theorem 6.15 that $\{\phi_n\}_0^\infty$ is an orthonormal basis for the space $\{f \in L^2 : \mathcal{F}^{-1}f(x) = 0 \text{ for } x < 0\}$.

7.3 Some applications

The Fourier transform is a useful tool for analyzing a great variety of problems in mathematics and the physical sciences. Underlying most of these applications is the following fundamental fact.

Suppose L is a linear operator on functions on $\mathbb{R}$ *that commutes with translations; that is, if* $L[f(x)] = g(x)$ *then* $L[f(x + s)] = g(x + s)$ *for any* $s \in \mathbb{R}$*. Then any exponential function* e^{ax} $(a \in \mathbb{C})$ *that belongs to the domain of L is an eigenfunction of L.*

The proof of this is very simple: let $f(x) = e^{ax}$ and $g = L[f]$. Then for any $s \in \mathbb{R}$,
$$g(x + s) = L[e^{a(x+s)}] = L[e^{as}e^{ax}] = e^{as}L[e^{ax}] = e^{as}g(x).$$

Setting $x = 0$, we find that $g(s) = g(0)e^{as}$ for all $s \in \mathbb{R}$; in other words, $g = Cf$ where $C = g(0)$. Thus $L[f] = Cf$.

Suppose in particular that the domain of L includes all the imaginary exponentials $e^{i\xi x}$, and let $h(\xi)$ be the eigenvalue for $e^{i\xi x}$; thus $L[e^{i\xi x}] = h(\xi)e^{i\xi x}$. If L satisfies some very mild continuity conditions, one can read off the action of L on a more or less arbitrary function f from the Fourier inversion formula. Indeed, that formula expresses f as a continuous superposition of the exponentials $e^{i\xi x}$,

$$f(x) = \frac{1}{2\pi}\int \widehat{f}(\xi)e^{i\xi x}\,d\xi,$$

and so

$$L[f](x) = \frac{1}{2\pi}\int \widehat{f}(\xi)L[e^{i\xi x}]\,d\xi = \frac{1}{2\pi}\int \widehat{f}(\xi)h(\xi)e^{i\xi x}\,d\xi.$$

(The continuity conditions on L are used here to justify treating the integral as if it were a finite sum.) Thus, in terms of the Fourier transform $\widehat{f}$, the action of L reduces to the simple algebraic operation of multiplication by the function h, $(L[f])\widehat{} = h\widehat{f}$, so passage from f to $\widehat{f}$ may simplify the analysis of L immensely. Alternatively, if h is the Fourier transform of a function H, we can express L as a convolution: $Lf = f * H$.

In order for these calculations to work according to the theory we have developed so far, f and H must be L^1 or L^2 functions. However, as we shall see in §9.4, it is possible to extend the domain of the Fourier transform to include much more general sorts of functions, and the present discussion then extends to the more general situation. In any case, at this point we are only giving an informal presentation of the ideas rather than precise results.

Perhaps the single most important class of operators L to which this analysis applies is the class of linear differential operators with constant coefficients: $L[f] = \sum_0^k c_j f^{(j)}$. Here, of course, we have $L[e^{i\xi x}] = \sum_0^k c_j(i\xi)^j e^{i\xi x}$; hence for any f, $(L[f])\widehat{}(\xi) = \sum_0^k c_j(i\xi)^j \widehat{f}(\xi)$, as we already know from Theorem 7.5(c). This fact is the basis for the use of the Fourier transform in solving differential equations, as we shall now demonstrate.

Partial differential equations

In Chapters 2 and 3 we saw how to solve certain boundary value problems for the heat, wave, and Laplace equations by means of Fourier series. We now use the Fourier transform to solve analogous problems on unbounded regions. The crux of the matter is Theorem 7.5(c), which says that the Fourier transform converts differentiation into a simple algebraic operation. By utilizing this fact we can reduce partial differential equations to easily solvable ordinary differential equations.

To begin with, consider heat flow in an infinitely long rod, given the initial temperature $f(x)$:

$$u_t = ku_{xx} \quad (-\infty < x < \infty), \qquad u(x,0) = f(x). \tag{7.19}$$

There are no boundary conditions for $t > 0$ because there is no boundary, but we shall assume (for the time being) that $u(x,t)$ and $f(x)$ vanish sufficiently rapidly as $x \to \pm\infty$ to be integrable over the whole line. Then we can apply the Fourier transform in x to convert (7.19) into

$$\frac{\partial \widehat{u}}{\partial t}(\xi,t) = -k\xi^2\widehat{u}(\xi,t), \qquad \widehat{u}(\xi,0) = \widehat{f}(\xi).$$

For each fixed ξ, this is a simple ordinary differential equation in t with an initial condition; its solution is

$$\widehat{u}(\xi,t) = \widehat{f}(\xi)e^{-k\xi^2 t}.$$

It remains to invert the Fourier transform, which can be done in either of two ways. The first is to apply the Fourier inversion theorem to obtain the **Fourier integral formula** for u:

$$u(x,t) = \frac{1}{2\pi}\int \widehat{f}(\xi)e^{-k\xi^2 t}e^{i\xi x}\,d\xi.$$

The second is to use Theorem 7.5(d) to obtain a formula for u as a convolution; this has the advantage of expressing u in terms of f rather than $\widehat{f}$. Namely, by formula (7.12) with $a = 1/2kt$, we see that the inverse Fourier transform of $e^{-k\xi^2 t}$ is

$$K_t(x) = \frac{1}{\sqrt{4\pi kt}}e^{-x^2/4kt}.$$

Therefore,

$$u(x,t) = f * K_t(x) = \frac{1}{\sqrt{4\pi kt}}\int f(y)e^{-(x-y)^2/4kt}\,dy. \tag{7.20}$$

Once we have this formula in hand, we can verify directly that it works. It is a simple exercise (Exercise 1 of §1.1) to check that $K_t(x)$ satisfies the heat equation, from which it follows by differentiating under the integral that $u(x,t)$ does also; and that $u(x,t) \to f(x)$ as $t \to 0$ (assuming, say, that f is continuous) follows from Theorem 7.3. Moreover, the hypothesis that $f \in L^1$ can be relaxed considerably. Since $K_t(x)$ decays very rapidly as $x \to \pm\infty$, the integral in (7.20) will converge as long as $f(x)$ grows less rapidly at infinity than any function $e^{\epsilon x^2}$ ($\epsilon > 0$), and an easy extension of the arguments just sketched shows that u still satisfies (7.19) in this case.

The physical interpretation of (7.20) is as follows. Imagine that the whole infinite rod starts out at temperature zero, and at time $t = 0$ a unit quantity of heat is injected at the origin. As t increases this heat spreads out along the rod, producing the temperature distribution $K_t(x)$. If, instead, the heat is injected at the point y, the resulting temperature distribution is $K_t(x-y)$. Now, in the problem (7.19), at time $t = 0$ there is an amount $f(y)\,dy$ of heat at the point y, which spreads out to give $K_t(x-y)f(y)\,dy$ at time $t > 0$. By the superposition principle, these temperatures can be added up to form (7.20).

Is (7.20) the *only* solution of (7.19)? Alas, the answer is *no,* for there exist nonzero solutions $v(x,t)$ of the heat equation with $v(x,0) = 0$. (The construction is rather complicated; see John [33] or Körner [34].) However, such functions $v(x,t)$ grow very rapidly as $x \to \pm\infty$, so they can be dismissed as physically unrealistic. What *is* true is that if the initial temperature $f(x)$ is bounded, then (7.20) is the only *bounded* solution of (7.19).

Let us now turn to the Dirichlet problem for a half-plane:

$$u_{xx} + u_{yy} = 0 \quad (x \in \mathbb{R},\ y > 0), \qquad u(x,0) = f(x). \tag{7.21}$$

Here again we must impose a boundedness condition to obtain uniqueness, and the reason is simple: If $u(x,y)$ satisfies (7.21), then so does $u(x,y) + y$. We therefore assume that f is bounded and (for the moment) integrable, and we seek a bounded solution of (7.21).

As with the heat equation, we begin by taking the Fourier transform in x:

$$-\xi^2\widehat{u}(\xi,y) + \widehat{u}(\xi,y)_{yy} = 0, \qquad \widehat{u}(\xi,0) = \widehat{f}(\xi).$$

This is an ordinary differential equation in y, and its general solution is

$$\widehat{u}(\xi,y) = C_1(\xi)e^{|\xi|y} + C_2(\xi)e^{-|\xi|y}, \qquad C_1(\xi) + C_2(\xi) = \widehat{f}(\xi).$$

Because of the boundedness requirement, we must reject the solution $e^{|\xi|y}$, so we take $C_1 = 0$ and $C_2 = \widehat{f}$. Hence, by Theorem 7.5(d), $u(x,y) = f * P_y(x)$ where $\widehat{P}_y(\xi) = e^{-|\xi|y}$, so by (7.13) (with y in place of a),

$$P_y(x) = \frac{y}{\pi(x^2+y^2)}.$$

Thus,

$$u(x,y) = f * P_y(x) = \int \frac{yf(x-t)}{\pi(t^2+y^2)}\,dt. \tag{7.22}$$

This is the **Poisson integral formula** for the solution of (7.21), and $P_y(x)$ is called the **Poisson kernel**. Since $P_y \in L^1$, (7.22) makes sense for any bounded f and defines a bounded function u:

$$|f| \le M \quad \Longrightarrow \quad |u(x,y)| \le M \int \frac{y}{\pi(t^2+y^2)}dt = M.$$

One can check directly that it satisfies Laplace's equation; and if f is (say) continuous, we have $u(x,y) \to f(x)$ as $y \to 0$ by Theorem 7.3.

The 1-dimensional wave equation can be solved by the same technique, leading to the solution found in Exercise 6, §1.1. We leave the details to the reader (Exercise 3).

Signal analysis

Let $f(t)$ represent the amplitude of a signal, perhaps a sound wave or an electromagnetic wave, at time t. The Fourier representation

$$f(t) = \frac{1}{2\pi} \int \widehat{f}(\omega) e^{i\omega t}\, d\omega, \qquad \widehat{f}(\omega) = \int f(t) e^{-i\omega t}\, dt$$

exhibits f as a continuous superposition of the simple periodic waves $e^{i\omega t}$ as ω ranges over all possible frequencies. This representation is absolutely basic in the analysis of signals in electrical engineering and information theory. Whole books can be, and have been, written on this subject; here we shall just present a couple of basic results to give the flavor of the ideas. For more extensive treatments we refer the reader to Bracewell [11], Dym-McKean [19], Papoulis [42], and Taylor [51].

In the first place, the power of a signal $f(t)$ is proportional to the square of the amplitude, $|f(t)|^2$, so the total energy of the signal is proportional to $\int |f(t)|^2\, dt$. Hence, the condition that the total energy be finite is just that $f \in L^2$.

Second, electrical systems can be mathematically modeled as operators L that transform an input signal f into an output signal $L[f]$. Many (but of course not all) such systems have a linear response, which means that the operator L is linear. Also, their action is generally unaffected by the passage of time (a given input signal produces the same response whether it was fed in yesterday or today), which means that L commutes with time translations. In this case, the general principles enunciated at the beginning of this section apply, and we see that L is described by

$$(L[f])^{\widehat{}} = h\widehat{f}, \quad \text{or} \quad L[f] = f * H,$$

where h is a certain complex-valued function and H is its inverse Fourier transform. h is called the **system function** and H is called the **impulse response**. ($H(t)$ is the output when the input is the Dirac δ-function; see Chapter 9.) If we write $h(\omega)$ in polar form as $h(\omega) = A(\omega)e^{i\theta(\omega)}$, $A(\omega)$ and $\theta(\omega)$ represent the amplitude and phase modulation due to the operator L at frequency ω.

Physical devices that work in real time must obey the law of causality, which means that if the input signal $f(t)$ is zero for $t < t_0$, then the output $g(t) = L[f](t)$ must also be zero for $t < t_0$. In other words,

$$f(t) = 0 \text{ for } t < t_0 \quad \Longrightarrow \quad f * H(t) = \int_{t_0}^{\infty} f(s)H(t-s)\, ds = 0 \text{ for } t < t_0.$$

The only way this can hold for all inputs f is to have $H(t-s) = 0$ when $t < t_0$ and $s > t_0$, and for this to be true for all t_0 we must have $H(t-s) = 0$ when $t - s < 0$, i.e., $H(t) = 0$ for $t < 0$.

This condition on H places rather severe restrictions on the system function h, and it often implies that certain desirable characteristics of an electrical system can be achieved only approximately. For example, one often wishes to filter out all frequencies outside some finite interval — say, outside the interval $[-\Omega, \Omega]$.

A device to accomplish this is called a *band-pass filter,* and the system function for an ideal band-pass filter would be the function $h(\omega)$ that equals 1 if $|\omega| \leq \Omega$ and equals 0 otherwise. But by a slight modification of formula (7.11), the corresponding impulse response $H(t)$ would be $H(t) = (\sin \Omega t)/2\pi t$, which does not obey the causality principle. It is, of course, the business of engineers to figure out ways to circumvent such difficulties!

Because of their importance in engineering as well as pure analysis, L^2 functions whose inverse Fourier transforms vanish on a half-line have been studied extensively. They are known as **Hardy functions**, and the space of Hardy functions is denoted by H^2. See Exercise 15 of §7.2 for the construction of a useful orthonormal basis for H^2, and Dym-McKean [19] for the physical interpretation of this basis and further information on H^2.

Let us now turn to a basic theorem of signal analysis that involves a neat interplay of Fourier transforms and Fourier series. Suppose f represents a physical signal that we are allowed to investigate by measuring its values at some sequence of times $t_1 < t_2 < \ldots$. How much information can we gain this way? Of course, for an arbitrary function $f(t)$, knowing a discrete set of values $f(t_1), f(t_2), \ldots$ tells us essentially nothing about the values of f at other points. However, if f is known to involve only certain frequencies, we can say quite a bit. To be precise, the signal f is called **band-limited** if it involves only frequencies smaller than some constant Ω, that is, if $\widehat{f}$ vanishes outside the finite interval $[-\Omega, \Omega]$. In this case, since $e^{i\omega t}$ does not change much on any interval of length $\Delta t \ll \omega^{-1}$, one has the intuitive feeling that $f(t + \Delta t)$ cannot differ much from $f(t)$ when $\Delta t \ll \Omega^{-1}$; hence one should pretty well know f once one knows the values $f(t_j)$ at a sequence $\{t_j\}$ of points with $t_{j+1} - t_j \approx \Omega^{-1}$.

This bit of folk wisdom can be made into an elegant and precise theorem by combining the techniques of Fourier series and Fourier integrals. Namely, suppose that $f \in L^2$ and f is band-limited. Then $\widehat{f} \in L^2$ and $\widehat{f}$ vanishes outside a finite interval, so $\widehat{f} \in L^1$. It follows that the Fourier inversion formula holds in the form (7.16). With this in mind, we have the following result.

The Sampling Theorem. *Suppose $f \in L^2$ and $\widehat{f}(\omega) = 0$ for $|\omega| \geq \Omega$. Then f is completely determined by its values at the points $t_n = n\pi/\Omega$, $n = 0, \pm 1, \pm 2, \ldots$. In fact,*

$$f(t) = \sum_{-\infty}^{\infty} f\left(\frac{n\pi}{\Omega}\right) \frac{\sin(\Omega t - n\pi)}{\Omega t - n\pi}. \tag{7.23}$$

Proof: Let us expand $\widehat{f}$ in a Fourier *series* on the interval $[-\Omega, \Omega]$, writing $-n$ in place of n for reasons of later convenience:

$$\widehat{f}(\omega) = \sum_{-\infty}^{\infty} c_{-n} e^{-in\pi\omega/\Omega} \qquad (|\omega| \leq \Omega).$$

The Fourier coefficients c_{-n} are given by

$$c_{-n} = \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} \widehat{f}(\omega) e^{in\pi\omega/\Omega} d\omega = \frac{1}{2\Omega} \int_{-\infty}^{\infty} \widehat{f}(\omega) e^{in\pi\omega/\Omega} d\omega = \frac{\pi}{\Omega} f\left(\frac{n\pi}{\Omega}\right).$$

Here we have used the fact that $\widehat{f}(\omega) = 0$ for $|\omega| > \Omega$ and the Fourier inversion formula (7.16). Using these two ingredients again, we obtain

$$f(t) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} \widehat{f}(\omega)e^{i\omega t}\,d\omega = \frac{1}{2\Omega}\int_{-\Omega}^{\Omega}\sum_{-\infty}^{\infty} f\left(\frac{n\pi}{\Omega}\right)e^{-in\pi\omega/\Omega}e^{i\omega t}\,d\omega$$

$$= \frac{1}{2\Omega}\sum_{-\infty}^{\infty} f\left(\frac{n\pi}{\Omega}\right)\frac{e^{i(\Omega t-n\pi)\omega/\Omega}}{i(\Omega t-n\pi)/\Omega}\bigg|_{-\Omega}^{\Omega} = \sum_{-\infty}^{\infty} f\left(\frac{n\pi}{\Omega}\right)\frac{\sin(\Omega t-n\pi)}{\Omega t-n\pi}.$$

(Termwise integration of the sum is permissible because the Fourier series of $\widehat{f}$ converges in $L^2(-\Omega,\Omega)$, and we are essentially taking the inner product of this series with $e^{i\omega t}$.) ∎

There is a dual formulation of this theorem for frequency sampling of time-limited functions. That is, suppose $f(t)$ vanishes for $|t| > L$. Then $\widehat{f}$ is determined by its values at the points $\omega_n = n\pi/L$ by the same formula (7.23) (with f replaced by $\widehat{f}$). The proof is essentially the same, because of the symmetry between f and $\widehat{f}$. The sampling theorem can also be modified to deal with signals whose Fourier transform vanishes outside an interval $[a,b]$ that is not centered at 0; see Exercise 7.

It is worth noting that the functions

$$s_n(t) = \frac{\sin(\Omega t-n\pi)}{\Omega t-n\pi} \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

form an orthogonal basis for the space of L^2 functions whose Fourier transforms vanish outside $[-\Omega,\Omega]$, and that the sampling formula (7.23) is merely the expansion of f with respect to this basis. Indeed, the calculations in the proof of the sampling theorem show that s_n is the inverse Fourier transform of the function

$$\widehat{s}_n(\omega) = \begin{cases} (\pi/\Omega)e^{-in\pi\omega/\Omega} & \text{if } |\omega| < \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

The assertion therefore follows from the Plancherel theorem and the fact that the functions $e^{-in\pi\omega/\Omega}$ constitute an orthogonal basis for $L^2(-\Omega,\Omega)$. Further discussion of the expansion (7.23) and related topics can be found in Higgins [29].

From a practical point of view, the expansion (7.23) has the disadvantage that it generally does not converge very rapidly, because the function $(\sin x)/x$ decays slowly as $x \to \infty$. A more rapidly convergent expansion for a function f can be obtained by *oversampling,* that is, by replacing the sequence of points $n\pi/\Omega$ at which f is sampled by a more closely spaced sequence $n\pi/\lambda\Omega$ ($\lambda > 1$). If this is done, one can replace $(\sin x)/x$ by a function that vanishes like x^{-2} as $x \to \infty$. The precise result is worked out in Exercise 8.

Heisenberg's inequality

It is impossible for a signal to be both band-limited and time-limited; that is, it is impossible for f and $\widehat{f}$ both to vanish outside a finite interval unless f is identically zero. Indeed, if $f \in L^2$ (say) and $\widehat{f}(\omega) = 0$ for $|\omega| > \Omega$, the integral

$$F(z) = \frac{1}{2\pi} \int e^{i\omega z} \widehat{f}(\omega)\, d\omega$$

makes sense for any complex number z; moreover, we can differentiate under the integral to see that $F(z)$ is analytic. Thus, f is the restriction to the real axis of the entire analytic function F, and in particular, f cannot vanish except at isolated points unless it vanishes identically. In exactly the same way, if $f \neq 0$ vanishes outside a finite interval then $\widehat{f}$ has only isolated zeros.

These facts are aspects of a general principle that says that f and $\widehat{f}$ cannot both be highly localized. That is, if f vanishes (or is very small) outside some small interval, then $\widehat{f}$ has to be quite "spread out," and vice versa. Another piece of supporting evidence for this idea is Theorem 7.5(b), which says in essence that composing f with a compression or expansion corresponds to composing $\widehat{f}$ with an expansion or compression, respectively. To obtain a precise quantitative result along these lines, we introduce the notion of the **dispersion** of f about the point a,

$$\Delta_a f = \int (x-a)^2 |f(x)|^2\, dx \Big/ \int |f(x)|^2\, dx.$$

$\Delta_a f$ is a measure of how much f fails to be concentrated near a. If f "lives near a," that is, if f is very small outside a small neighborhood of a, then the factor of $(x-a)^2$ will make the numerator of $\Delta_a f$ small in comparison to the denominator, whereas if f "lives far away from a," the same factor will make the numerator large in comparison to the denominator. The following theorem therefore says that f and $\widehat{f}$ cannot both be concentrated near single points.

Heisenberg's Inequality. *For any $f \in L^2$,*

$$(\Delta_a f)(\Delta_\alpha \widehat{f}) \geq \frac{1}{4} \quad \text{for all } a, \alpha \in \mathbf{R}. \tag{7.24}$$

Proof: For technical convenience we shall assume that f is continuous and piecewise smooth, and that the functions $xf(x)$ and $f'(x)$ are in L^2. (The smoothness assumption can be removed by an additional limiting argument; see Dym-McKean [19]. If $xf(x)$ is not in L^2 then $\Delta_a f = \infty$, whereas if $f'(x)$ is not in L^2 then $\Delta_\alpha \widehat{f} = \infty$, as the calculations below will show; in either case, (7.24) is trivially true.) Let us first consider the case $a = \alpha = 0$. By integration by parts, we have

$$\int_A^B x\overline{f(x)} f'(x)\, dx = x|f(x)|^2 \Big|_A^B - \int_A^B \left(|f(x)|^2 + x f(x) \overline{f'(x)} \right) dx,$$

or

$$\int_A^B |f(x)|^2 dx = -2\,\mathrm{Re}\int_A^B \overline{xf(x)}f'(x)\,dx + x|f(x)|^2\Big|_A^B$$
$$= -2\,\mathrm{Re}\int_A^B \overline{g(x)}f'(x)\,dx + x|f(x)|^2\Big|_A^B.$$

Since f, g, and f' are in L^2, the limits of the integrals in this equation as $A \to -\infty$ and $B \to \infty$ exist. Hence, so do the limits of $A|f(A)|^2$ and $B|f(B)|^2$, and these limits must be zero. (Otherwise, $|f(x)| \sim |x|^{-1/2}$ for large x, and f would not be in L^2.) We can therefore let $A \to -\infty$ and $B \to \infty$ to obtain

$$\int |f(x)|^2 dx = -2\,\mathrm{Re}\int \overline{xf(x)}f'(x)\,dx.$$

By the Cauchy-Schwarz inequality, then,

$$\left(\int |f(x)|^2 dx\right)^2 \le 4\left(\int x^2|f(x)|^2 dx\right)\left(\int |f'(x)|^2 dx\right). \tag{7.25}$$

But by the Plancherel theorem, $\int |f|^2 = (2\pi)^{-1}\int |\widehat{f}|^2$ and

$$\int |f'(x)|^2 dx = \frac{1}{2\pi}\int |[f']^{\wedge}(\xi)|^2\,d\xi = \frac{1}{2\pi}\int \xi^2|\widehat{f}(\xi)|^2\,d\xi.$$

Hence (7.25) can be rewritten as

$$\left(\int |f(x)|^2\,dx\right)\left(\int |\widehat{f}(\xi)|^2\,d\xi\right) \le 4\left(\int x^2|f(x)|^2\,dx\right)\left(\int \xi^2|\widehat{f}(\xi)|^2\,d\xi\right),$$

which is (7.24) with $a = \alpha = 0$.

The general case is easily reduced to this one by a change of variable. Namely, given a and α, let

$$F(x) = e^{-i\alpha x}f(x+a).$$

It is easily verified that F satisfies the hypotheses of the theorem and that $\Delta_a f = \Delta_0 F$ and $\Delta_\alpha \widehat{f} = \Delta_0 \widehat{F}$. (See Exercise 9.) We can therefore apply the preceding argument to F to conclude that

$$(\Delta_a f)(\Delta_\alpha \widehat{f}) = (\Delta_0 F)(\Delta_0 \widehat{f}) \ge \frac{1}{4}. \qquad \blacksquare$$

Quantum mechanics

The Fourier transform is an essential tool for the quantum-mechanical description of nature. It would take us too far afield to explain the physics here; but for those who have the necessary physical background, we present a brief discussion of the mathematical formalism. For more details, see Messiah [39] or Landau-Lifshitz [35].

In quantum mechanics, a particle such as an electron that moves along the x-axis is described by a "wave function" $f(x)$, which is a complex-valued L^2 function such that $\|f\| = 1$. $|f(x)|^2$ is interpreted as the probability density that the particle will be found at position x; that is, $\int_a^b |f(x)|^2\,dx$ is the probability that the particle will be found in the interval $[a, b]$. (The condition $\|f\| = 1$ guarantees that the total probability is 1.)

The Fourier transform of the wave function f essentially gives the probability density for the momentum of the particle. More precisely, we define a modified Fourier transform $\widetilde{f}$ by

$$\widetilde{f}(p) = \frac{1}{\sqrt{2\pi\hbar}}\widehat{f}\left(\frac{p}{\hbar}\right) = \frac{1}{\sqrt{2\pi\hbar}}\int f(x)e^{-ixp/\hbar}\,dx,$$

where $\hbar$ is Planck's constant. Then the Plancherel theorem implies that

$$\int |\widetilde{f}(p)|^2\,dp = \frac{1}{2\pi\hbar}\int |\widehat{f}(\hbar^{-1}p)|^2\,dp = \frac{1}{2\pi}\int |\widehat{f}(\xi)|^2\,d\xi = \int |f(x)|^2\,dx = 1.$$

Thus $|\widetilde{f}(p)|^2$ can be interpreted as a probability density, and it is the probability density for momentum.

Similar considerations apply to particles moving in 3-space. One merely has to use the 3-dimensional version of the Fourier transform; see §7.5.

Heisenberg's inequality is a precise formulation of the position-momentum uncertainty principle. The numbers $\Delta_a f$ and $\Delta_\alpha \widetilde{f}$ are measures of how much the probability distributions $|f|^2$ and $|\widetilde{f}|^2$ are spread out away from the points a and α. (If we take a and α to be the mean values of these distributions, $\Delta_a f$ and $\Delta_\alpha \widetilde{f}$ are their variances.) A simple change of variable shows that $\Delta_\alpha \widetilde{f} = \hbar^2 \Delta_{\alpha/\hbar} \widehat{f}$, so Heisenberg's inequality says that

$$(\Delta_a f)(\Delta_\alpha \widetilde{f}) \geq \hbar^2/4.$$

The uncertainty principle is often cited as one of the mysteries of quantum mechanics, but the inverse relationship between the spatial or temporal localization of a wave and the localization of its frequency spectrum is a general phenomenon that pertains to waves of any sort. What is strange about the quantum world is that particles behave in some respects like waves.

Other applications

The Fourier transform is a ubiquitous tool in many fields of science as well as in pure mathematics. For discussions of some of its other applications we refer the reader to Bracewell [11], Dym-McKean [19], Körner [34], Papoulis [42], and Walker [53], [54].

EXERCISES

1. Use the Fourier transform to find a solution of the ordinary differential equation $u'' - u + 2g(x) = 0$ where $g \in L^1$. (The solution obtained this way is the one that vanishes at $\pm\infty$. What is the general solution?)

2. Use the Fourier transform to derive the solution

$$u(x,t) = f * K_t(x) + \int_{-\infty}^{\infty} \int_0^t G(y,s) K_{t-s}(x-y)\, ds\, dy$$

of the inhomogeneous heat equation $u_t = ku_{xx} + G(x,t)$ with initial condition $u(x,0) = f(x)$, where K_t is as in (7.20). (Observe that if we set $K_t(x) = 0$ and $G(x,t) = 0$ for $t \leq 0$, then the second term of u is the convolution of G with K in the x *and* t variables.)

3. Consider the wave equation $u_{tt} = c^2 u_{xx}$ with initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$.

   a. Assuming that all the Fourier transforms in question exist, show that

   $$\widehat{u}(\xi,t) = \widehat{f}(\xi) \cos ct\xi + \widehat{g}(\xi)(c\xi)^{-1} \sin ct\xi.$$

   b. Invert the Fourier transform to obtain d'Alembert's formula for u (Exercise 6, §1.1). (Hint: For the first term, write $\cos ct\xi = \frac{1}{2}(e^{ict\xi} + e^{-ict\xi})$ and use Theorem 7.5(a); for the second one, cf. formula (7.11).)

4. Solve the Dirichlet problem in an infinite strip: $u_{xx} + u_{yy} = 0$ for $x \in \mathbf{R}$ and $0 < y < b$, $u(x,0) = f(x)$, $u(x,b) = g(x)$. (Hint: First do the case $f = 0$. The case $g = 0$ reduces to this one by the substitution $y \to b - y$, and the general case is obtained by superposition. Exercise 10, §7.2 is useful.)

5. Let S be the infinite cylinder of radius a, given in cylindrical coordinates (r, θ, z) by the equation $r = a$. Find the electrostatic potential u inside S if the portion of S with $|z| < l$ is held at potential 1 and the rest of S is held at potential 0. (u is clearly independent of θ, so the problem to be solved is $u_{rr} + r^{-1}u_r + u_{zz} = 0$ inside S, $u(a,z) = 1$ if $|z| < l$ and $u(a,z) = 0$ otherwise. Use the Fourier transform in z, and express the answer as a Fourier integral.)

6. Suppose $f \in L^2$ represents a signal. Show that the best approximation to f in the L^2 norm among all signals that are band-limited to the interval $[-\Omega, \Omega]$ is $g_0(t) = (2\pi)^{-1} \int_{-\Omega}^{\Omega} \widehat{f}(\omega) e^{i\omega t}\, d\omega$. That is, show that $\|g_0 - f\| \leq \|g - f\|$ for all g such that $\widehat{g}(\omega) = 0$ for $|\omega| > \Omega$. (Use the Plancherel theorem, and cf. Theorem 3.8 of §3.4.)

7. State and prove a version of the sampling theorem for signals whose Fourier transforms vanish outside an interval $[a,b]$. (A simple-minded answer to this problem is the following: If, say, $0 < a < b$, then $[a,b] \subset [-b,b]$, and one can apply the sampling theorem with $\Omega = b$. But this gives a formula for f in terms of its values at the points $n\pi/b$, whereas the optimal theorem involves the more widely spaced points $2n\pi/(b-a)$. Hint: If $\widehat{f}$ vanishes outside $[a,b]$, consider $g(t) = e^{-i(b-a)t/2} f(t)$.)

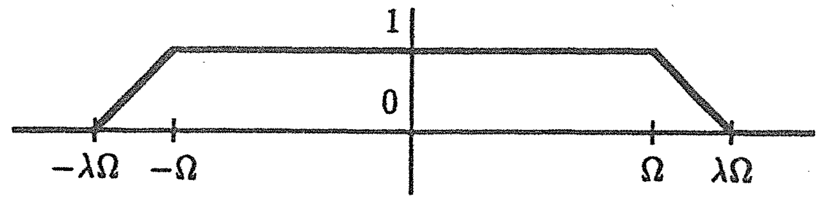
8. Suppose $f \in L^2(\mathbf{R})$, $\widehat{f}(\omega) = 0$ for $|\omega| > \Omega$, and $\lambda > 1$.

   a. As in the proof of the sampling theorem, show that

   $$\widehat{f}(\omega) = \frac{\pi}{\lambda\Omega} \sum_{-\infty}^{\infty} f\left(\frac{n\pi}{\lambda\Omega}\right) e^{-in\pi\omega/\lambda\Omega} \quad \text{for} \quad |\omega| \leq \lambda\Omega.$$

b. Let $\widehat{g}_\lambda$ be the piecewise linear function sketched below. Show that the inverse Fourier transform of $\widehat{g}_\lambda$ is

$$g_\lambda(t) = \frac{\cos \Omega t - \cos \lambda\Omega t}{\pi(\lambda - 1)\Omega t^2}.$$



c. Observe that $\widehat{f} = \widehat{g}_\lambda \widehat{f}$. By substituting the expansion in part (a) into the Fourier inversion formula, show that

$$f(t) = \frac{1}{2\pi}\int_{-\lambda\Omega}^{\lambda\Omega} \widehat{f}(\omega)\widehat{g}_\lambda(\omega)e^{i\omega t}\,d\omega = \frac{\pi}{\lambda\Omega}\sum_{-\infty}^{\infty} f\left(\frac{n\pi}{\lambda\Omega}\right) g_\lambda\left(t - \frac{n\pi}{\lambda\Omega}\right).$$

This gives a sampling formula for f in which the basic functions $g_\lambda(t)$ decay like t^{-2} at infinity.

9. Suppose that f satisfies the hypotheses of Heisenberg's inequality, and let $F(x) = e^{-i\alpha x} f(x + a)$.
   a. Show that $\Delta_a f = \Delta_0 F$.
   b. Show that $\widehat{F}(\xi) = e^{ia(\xi+\alpha)}\widehat{f}(\xi + \alpha)$ and thence that $\Delta_\alpha \widehat{f} = \Delta_0 \widehat{F}$.
10. Show that Heisenberg's inequality $(\Delta_0 f)(\Delta_0 \widehat{f}) \geq \frac{1}{4}$ is an equality if and only if $f' + cf = 0$ where c is a real constant, and hence show that the functions that minimize the uncertainty product $(\Delta_0 f)(\Delta_0 \widehat{f})$ are precisely those of the form $f(x) = Ce^{-cx^2/2}$ for some $c > 0$. (Hint: Examine the proof of Heisenberg's inequality and recall that the Cauchy-Schwarz inequality $|\int f g| \leq \|f\|\,\|g\|$ is an equality if and only if f and g are scalar multiples of one another.) What are the minimizing functions for the uncertainty product $(\Delta_a f)(\Delta_\alpha \widehat{f})$ for general a, α? (Cf. Exercise 9.)

7.4 Fourier transforms and Sturm-Liouville problems

In §7.3 we solved some boundary value problems by applying the Fourier transform. The same results could have been obtained from a slightly different point of view, starting with separation of variables. For example, for functions $u(x,t) = X(x)T(t)$ the heat equation $u_t = ku_{xx}$ separates into the ordinary differential equations $T' = -\xi^2 kT$ and $X'' + \xi^2 X = 0$ where ξ^2 is the separation constant. Solution of these equations leads to the products $u(x,t) = e^{-\xi^2 kt}e^{i\xi x}$ and hence to their continuous superpositions

$$u(x,t) = \int c(\xi)e^{-\xi^2 kt}e^{i\xi x}\,d\xi.$$

If the initial condition is $u(x, 0) = f(x)$, one sees from the Fourier inversion formula that $c(\xi) = (2\pi)^{-1}\widehat{f}(\xi)$, which leads to the solution (7.20).

What is at issue here is the singular Sturm-Liouville problem

$$X''(x) + \xi^2 X(x) = 0, \qquad -\infty < x < \infty.$$

The general solution of this equation is $c_1 e^{i\xi x} + c_2 e^{-i\xi x}$ for $\xi \neq 0$ or $c_1 + c_2 x$ for $\xi = 0$. None of these functions is in $L^2(\mathbf{R})$ except for the trivial case $c_1 = c_2 = 0$, so there is no possibility of finding an orthonormal basis of eigenfunctions. Instead, the expansion of an arbitrary $f \in L^2(\mathbf{R})$ in terms of these eigenfunctions is given by the Fourier inversion formula

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\xi)e^{i\xi x}\,d\xi = \frac{1}{2\pi}\int_0^{\infty}\left[\widehat{f}(\xi)e^{i\xi x} + \widehat{f}(-\xi)e^{-i\xi x}\right]d\xi \tag{7.26}$$

(with the integral suitably interpreted).

The reader may wonder what justification we have for restricting attention to real values of ξ in this situation. The practical answer is that (7.26) works, so no nonreal values of ξ are needed. A rather vague but more satisfying reason, which applies also to other problems of this sort, is the following. When $\operatorname{Im}\xi \neq 0$, $e^{i\xi x}$ blows up exponentially as $x \to \infty$ or $x \to -\infty$, so it fails so miserably to be in L^2 that it cannot be of any pertinence to an L^2 problem. But when ξ is real, $e^{i\xi x}$ is close enough to being in L^2 that it can contribute to an eigenfunction expansion by an infinitesimal amount, as in (7.26).

Let us now consider two singular Sturm-Liouville problems pertaining to functions on the half-line $[0, \infty)$:

$$X''(x) + \xi^2 X(x) = 0 \quad (0 < x < \infty), \qquad X'(0) = 0; \tag{7.27a}$$

$$X''(x) + \xi^2 X(x) = 0 \quad (0 < x < \infty), \qquad X(0) = 0. \tag{7.27b}$$

In (7.27a) the solutions that satisfy the boundary condition are multiples of $\cos \xi x$, whereas in (7.27b) they are multiples of $\sin \xi x$. Again, none of these functions are in $L^2(0, \infty)$, so there is no orthonormal basis of eigenfunctions. Instead, we can hope to find expansion formulas similar to (7.26), namely,

$$f(x) = \int_0^{\infty} a(\xi)\cos \xi x\,d\xi, \qquad f(x) = \int_0^{\infty} b(\xi)\sin \xi x\,d\xi$$

for $f \in L^2(0, \infty)$. In fact, such formulas can easily be derived from the Fourier transform by the same device by which we obtained Fourier sine and cosine series on $[0, \pi]$ from Fourier series on $[-\pi, \pi]$, namely, consideration of the even and odd extensions of f to $\mathbf{R}$.

Indeed, if $f \in L^1(\mathbf{R})$ and f is even, then

$$\widehat{f}(\xi) = \int f(x)(\cos \xi x - i \sin \xi x)\,dx = \int f(x)\cos \xi x\,dx = 2\int_0^{\infty} f(x)\cos \xi x\,dx.$$

From this it is clear that $\widehat{f}$ is also even, so the inversion formula (suitably interpreted as a limit as in Theorem 7.6) becomes

$$f(x) = \frac{1}{2\pi}\int \widehat{f}(\xi)(\cos\xi x + i\sin\xi x)\,d\xi = \frac{1}{\pi}\int_0^\infty \widehat{f}(\xi)\cos\xi x\,d\xi.$$

In the same way, we see that if f is odd, then so is $\widehat{f}$, and

$$\widehat{f}(\xi) = -2i\int_0^\infty f(x)\sin\xi x\,dx, \qquad f(x) = \frac{i}{\pi}\int_0^\infty \widehat{f}(\xi)\sin\xi x\,d\xi.$$

These formulas involve only the values of f and $\widehat{f}$ on $[0,\infty)$, so we can use them on functions that are initially defined only on $[0,\infty)$. This suggests the following definitions.

Suppose now that $f \in L^1(0,\infty)$. We define the **Fourier cosine transform** and **Fourier sine transform** of f to be the functions $\mathcal{F}_c[f]$ and $\mathcal{F}_s[f]$ on $[0,\infty)$ defined by

$$\mathcal{F}_c[f](\xi) = \int_0^\infty f(x)\cos\xi x\,dx \quad \text{and} \quad \mathcal{F}_s[f](\xi) = \int_0^\infty f(x)\sin\xi x\,dx.$$

Thus, if f_{even} and f_{odd} are the even and odd extensions of f to $\mathbb{R}$, $\mathcal{F}_c[f]$ and $\mathcal{F}_s[f]$ are the restrictions to $[0,\infty)$ of $\frac{1}{2}\widehat{f}_{\text{even}}$ and $\frac{1}{2}i\widehat{f}_{\text{odd}}$. The inversion formulas therefore become

$$f(x) = \frac{2}{\pi}\int_0^\infty \mathcal{F}_c[f](\xi)\cos\xi x\,d\xi = \frac{2}{\pi}\int_0^\infty \mathcal{F}_s[f](\xi)\sin\xi x\,d\xi, \tag{7.28}$$

giving the desired expansions of f in terms of cosines and sines. Here, of course, the integrals must be interpreted suitably. For example, if f is piecewise continuous we have

$$\lim_{\epsilon\to 0}\frac{2}{\pi}\int_0^\infty e^{-\epsilon^2\xi^2/2}\mathcal{F}_c[f](\xi)\cos\xi x\,d\xi = \tfrac{1}{2}\big[f(x-)+f(x+)\big].$$

The Parseval formula for $\mathcal{F}_c$ is obtained as follows:

$$\begin{aligned}\int_0^\infty |\mathcal{F}_c[f](\xi)|^2\,d\xi &= \frac{1}{4}\int_0^\infty |\widehat{f}_{\text{even}}(\xi)|^2\,d\xi = \frac{1}{8}\int_{-\infty}^\infty |\widehat{f}_{\text{even}}(\xi)|^2\,d\xi \\ &= \frac{\pi}{4}\int_{-\infty}^\infty |f_{\text{even}}(x)|^2\,dx = \frac{\pi}{2}\int_0^\infty |f(x)|^2\,dx,\end{aligned}$$

and similarly for $\mathcal{F}_s$. From this one obtains the analogue of the Plancherel theorem: *$\mathcal{F}_c$ and $\mathcal{F}_s$ extend to maps from $L^2(0,\infty)$ onto itself that satisfy*

$$\|\mathcal{F}_c[f]\|^2 = \|\mathcal{F}_s[f]\|^2 = \frac{\pi}{2}\|f\|^2.$$

This fact, in conjunction with (7.28), gives the eigenfunction expansions for $L^2(0,\infty)$ associated to the Sturm-Liouville problems (7.27).

$\mathscr{F}_c$ and $\mathscr{F}_s$ have operational properties similar to the ones for the ordinary Fourier transform given in Theorem 7.5, but they are not quite as simple. For example, here is how $\mathscr{F}_c$ interacts with convolutions. Suppose f and g are (say) bounded and integrable on $[0,\infty)$, and let F and G be their even extensions to $\mathbb{R}$. It is easily verified that $F * G$ is also even, so the convolution formula $(F*G)\widehat{\ } = \widehat{F}\widehat{G}$ turns into $\mathscr{F}_c[h] = \mathscr{F}_c[f]\cdot\mathscr{F}_c[g]$ where $2h$ is the restriction of $F*G$ to $[0,\infty)$. (The factor of 2 is there because $\mathscr{F}_c[f]$ is the restriction of $\frac{1}{2}\widehat{f}$, rather than $\widehat{f}$, to $[0,\infty)$.) We can evaluate h directly in terms of f and g, as follows: We have

$$
\begin{aligned}
F * G(x) &= \int_{-\infty}^{0} F(y)G(x-y)\,dy + \int_{0}^{\infty} F(y)G(x-y)\,dy \\
&= \int_{0}^{\infty} F(y)G(x+y)\,dy + \int_{0}^{\infty} F(y)G(|x-y|)\,dy,
\end{aligned}
$$

where we have substituted $-y$ for y in the first integral and used the evenness of F and G. When $x \geq 0$, the arguments y, $x+y$, and $|x-y|$ are all positive, so in this last expression we can replace F and G by f and g. In short, we have

$$
\mathscr{F}_c[f]\cdot\mathscr{F}_c[g] = \mathscr{F}_c[h], \qquad h(x) = \frac{1}{2}\int_0^\infty f(y)\Big[g(x+y) + g(|x-y|)\Big]\,dy. \tag{7.29}
$$

Similar formulas for $\mathscr{F}_c[f]\cdot\mathscr{F}_s[g]$ and $\mathscr{F}_s[f]\cdot\mathscr{F}_s[g]$ exist; see Exercise 2. Also, see Exercise 3 for the interaction of $\mathscr{F}_c$ and $\mathscr{F}_s$ with derivatives.

Example. Consider heat flow in a semi-infinite rod insulated along its length and at the end:

$$
u_t = ku_{xx} \quad \text{for } x,t>0, \qquad u_x(0,t)=0, \qquad u(x,0)=f(x).
$$

Separation of variables in the differential equation together with the boundary condition $u_x(0,t)=0$ leads to the product solutions $e^{-\xi^2 kt}\cos\xi x$ and hence to their superpositions

$$
u(x,t) = \int_0^\infty c(\xi)e^{-\xi^2 kt}\cos\xi x\,d\xi.
$$

Setting $t=0$ and applying (7.28), we see that $c(\xi) = (2/\pi)\mathscr{F}_c[f](\xi)$, or

$$
u(x,t) = \frac{2}{\pi}\int_0^\infty \mathscr{F}_c[f](\xi)e^{\xi^2 kt}\cos\xi x\,d\xi.
$$

This is the **Fourier integral** formula for the solution. A formula that gives the solution directly in terms of f instead of $\mathscr{F}_c[f]$ may be derived from it as follows. By a simple modification of (7.12),

$$
e^{-\nu^2 kt} = \mathscr{F}_c[g_t](\nu) \quad \text{where } g_t(x) = \frac{1}{\sqrt{\pi kt}}e^{-x^2/4kt},
$$

so by (7.29),

$$u(x,t) = \frac{1}{2\sqrt{\pi kt}} \int_0^\infty f(y)\left[e^{-(x-y)^2/4kt} + e^{-(x+y)^2/4kt}\right] dy.$$

Of course this is nothing but the solution $u(x,t) = F * K_t(x)$ derived in §7.3, where F is the even extension of f to $\mathbb{R}$. The reader should take a minute or two to figure out why the latter construction gives the right answer.

We have shown how to derive Fourier cosine and sine transforms from the ordinary Fourier transform; one can also go the other way. Indeed, any function f on $\mathbb{R}$ is the sum of an even function and an odd function:

$$f = f_0 + f_1 \quad \text{where } f_0(x) = \frac{f(x)+f(-x)}{2}, \; f_1(x) = \frac{f(x)-f(-x)}{2}.$$

Since the even and odd parts of $e^{-i\xi x}$ are $\cos \xi x$ and $-i \sin \xi x$, we then have

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f_0(x) \cos \xi x \, dx - i \int_{\infty}^{\infty} f_1(x) \sin \xi x \, dx = 2\mathscr{F}_c[f_0](\xi) - 2i\mathscr{F}_s[f_1](\xi).$$

This observation is helpful for using tables of Fourier transforms such as Erdélyi et al. [22], where most of the entries are given in terms of cosine and sine transforms.

EXERCISES

1. Compute the following transforms, where $k > 0$.

   a. $\mathscr{F}_s[e^{-kx}]$ b. $\mathscr{F}_c[e^{-kx}]$ c. $\mathscr{F}_c\left[(1+x)e^{-x}\right]$ d. $\mathscr{F}_s[xe^{-x}]$

2. Let f and g be in $L^1(0,\infty)$. Show that $\mathscr{F}_s[f]\mathscr{F}_c[g] = \mathscr{F}_s[h]$ where

   $$h(x) = \int_0^\infty f(y)\frac{g(|x-y|) - g(x+y)}{2}\, dy,$$

   and that $\mathscr{F}_s[f]\mathscr{F}_s[g] = \mathscr{F}_c[H]$ where

   $$H(x) = \int_0^\infty f(y)\frac{\operatorname{sgn}(x-y)g(|x-y|) - g(x+y)}{2}\, dy,$$

   with $\operatorname{sgn}(t) = 1$ if $t > 0$ and $\operatorname{sgn}(t) = -1$ if $t < 0$.
3. Suppose that f is continuous and piecewise smooth and that f and f' are in $L^1(0,\infty)$. Show that

   $$\mathscr{F}_c[f'](\xi) = \xi\mathscr{F}_s[f](\xi) - f(0), \qquad \mathscr{F}_s[f'](\xi) = -\xi\mathscr{F}_c[f](\xi).$$

4. Solve the heat equation $u_t = ku_{xx}$ on the half-line $x > 0$ with boundary conditions $u(x,0) = f(x)$ and $u(0,t) = 0$. (Exercise 2 is useful.) Then solve

the inhomogeneous equation $u_t = ku_{xx} + G(x,t)$ with the same boundary conditions. (Cf. Exercise 1, §7.3.)

5. Solve the Dirichlet problem in the first quadrant: $u_{xx} + u_{yy} = 0$ for $x, y > 0$, $u(x,0) = f(x)$, $u(0,y) = g(y)$. (Hint: Consider the special cases $f = 0$ and $g = 0$. Use Exercise 2.)
6. Solve Laplace's equation $u_{xx} + u_{yy} = 0$ in the semi-infinite strip $x > 0$, $0 < y < 1$ with boundary conditions $u_x(0,y) = 0$, $u_y(x,0) = 0$, $u(x,1) = e^{-x}$. Express the answer as a Fourier integral. (Use Exercise 1.)
7. Do Exercise 6 with the first two boundary conditions replaced by $u(0,y) = 0$ and $u(x,0) = 0$.
8. Find the steady-state temperature in a plate occupying the semi-infinite strip $x > 0$, $0 < y < 1$ if the edges $y = 0$ and $x = 0$ are insulated, the edge $y = 1$ is maintained at temperature 1 for $x < c$ and at temperature zero for $x > c$, and the faces of the plate lose heat to the surroundings according to Newton's law with proportionality constant h. That is, solve

$$u_{xx} + u_{yy} - hu = 0, \quad u_x(0,y) = u_y(x,0) = 0,$$
$$u(x,1) = 1 \text{ if } x < c, \quad u(x,1) = 0 \text{ if } x \geq c.$$

Express the answer as a Fourier integral.

9. Consider the singular Sturm-Liouville problem

$$\left(rf'(r)\right)' + \lambda r^{-1} f(r) = 0 \quad \text{for } 0 < r < 1, \qquad f(1) = 0. \qquad (*)$$

   a. Show that the substitution $r = e^{-x}$, $g(x) = f(e^{-x})$ converts $(*)$ into the problem $g'' + \lambda g = 0$ for $0 < x < \infty$, $g(0) = 0$. Put this together with the Fourier sine transform to derive the "eigenfunction expansion" of a function $f \in L^2_{1/r}(0,1)$ associated to $(*)$.
   b. Use the result of (a) to solve Exercise 8b in §4.4:

$$\nabla^2 u = 0 \quad \text{in } \left\{(r,\theta) : 0 \leq r \leq 1,\ 0 \leq \theta \leq \beta\right\},$$
$$u(r,0) = f(r), \qquad u(r,\beta) = g(r), \qquad u(1,\theta) = 0.$$

7.5 Multivariable convolutions and Fourier transforms

In this section we consider functions of n real variables, that is, functions on the space $\mathbb{R}^n$ of n-tuples of real numbers. The notation for points in $\mathbb{R}^n$ will be $\mathbf{x} = (x_1, \ldots, x_n)$. We denote by $\mathbf{x} \cdot \mathbf{y}$ and $|\mathbf{x}|$ the usual dot product and norm on $\mathbb{R}^n$:

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n,$$
$$|\mathbf{x}| = (\mathbf{x} \cdot \mathbf{x})^{1/2} = \left(x_1^2 + \cdots + x_n^2\right)^{1/2}.$$